Foundations for Quantum Computation and Quantum Information

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Abstract

During the academic year of 2018-2019 I became critically aware of the importance of quantum information in Physics. I mean I thought it was cool before, because of all of the great things quantum computing brings with it, but the far reaching ideas about the foundations of the universe we inhabit it brings with it is something I did not expect. Ideas of this sort, making use of quantum information as their basis, are relativistic quantum information, quantum metrology and entanglement space time to mention a few. These ideas shake me to my core and I want to see where they can take me.
To this end I'm reading a number of things on the subject. These are the notes I take whilst I read.

1 Tensor Products & Entanglement

1.1 Mathematical & Physical Motivations for the Tensor Product

A Conceptual Stance To describe singular quantum states we use singular vector spaces composed of their own orthonormal eigenbasis but what of multiple particles? Specifically, let us at first limit our view to two particles.
Using the introduction that is given by Zwiebach in [4], consider a Particle 1 whose phase space is described by a complex inner product space V with operators $T_1, T_2, \dots$ and Particle 2 with similar space W and operators $S_1, S_2, \dots$ .To describe the two particles in conjunction to one another one might be inclined to do as we do in mathematics with Cartesian Products and leading to ordered pairs $(v, w) \in V \times W$. Now we're not quite there yet what we wish to define is a new vector space $V \otimes W$ with elements denoted $v \otimes w$ which start out in $V \times W$ but then are surjectively mapped onto $V \otimes W$ using the two physically motivated rules, which in mathematics refers to bilinear composition.

1. Scaling either states of the states v, w is equivalent to scaling both states.

$$(av) \otimes w = v \otimes (aw) = a\,(v \otimes w)\,, \;\; a \in \mathbb{C}.$$

2. If one of the two states is a superposition state so must be the tensor product of the two states. From a mathematical perspective this refers to the distributive nature of the tensor product over addition. From a physical point of view we will see that this means that if one of two particles is in a superposition state then the two-particle system is in a superposition state also. The astute reader will liken this property to that innate of homomorphisms.

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \qquad\qquad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2.$$

For V with basis $\{v_1, v_2, \dots v_n\}$ and W with basis $\{w_1, w_2, \dots w_m\}$ their tensor product $V \otimes W$ would be built up from basis elements $v_i \otimes w_j$ giving a basis looking $\{v_1 \otimes w_1, \dots, v_1 \otimes w_m, v_2 \otimes w_1, \dots, v_2 \otimes w_m, \dots\dots v_n \otimes w_1, \dots, v_n \otimes w_m\}$ visibly showing that

$$\dim V \otimes W = \dim V \times \dim W.$$

For the sake of pedantry consider $a \in V$ and $b \in W$. Their tensor product $a \otimes b \in V \otimes W$ is clearly a linear combination of the basis $v_i \otimes f_j$ since

$$a \otimes b = \sum_{i=1}^{n} (a \cdot v_i)\, v_i \otimes \sum_{j=1}^{m} (b \cdot w_j)\, w_j = \sum_{i,j} (a \cdot v_i)\, (b \cdot w_j)\, v_i \otimes w_j$$

clearly showing that $a \otimes b$ is a linear combination of $v_i \otimes w_j$.

An Operator defined to act on $V \otimes W$ must itself be a tensor product of operators defined on the subspaces V and W respectively. So for two operators $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$, their tensor product $T \otimes S \in \mathcal{L}(V \otimes W)$. Operators in a tensor product space act on elements of this space, themselves tensor products in the following logical way.

$$T \otimes S\,(v \otimes w) = Tv \otimes Sw$$

where each operator acts on the subspace upon which it is defined and the tensor product of the new vectors in V and W is used to calculate the new tensor product. A tensor product operator which acts on only one of two component states in a bipartite state maybe constructed using the identity operator $\mathcal{I}$. Such operators are called *Upgraded* operators in the sense that if T is an operator which works in V then $T \otimes \mathcal{I}$ is the upgraded version of the operator which does the same thing as T but in the tensor space $V \otimes W$.

$$H_1 \otimes H_2 \neq H_1 \otimes \mathcal{I} + \mathcal{I} \otimes H_2.$$

Let H represent a hamiltonian in a phase space of one of two systems. On the right we are thinking of both systems together but there is no mixing between the two system and hence the sum of upgraded operators. On the left we have mixing.

A Pragmatic and Literal Stance with Examples Having understood the conceptual aspects of the tensor product we now ground it in reality by understanding how the tensor product is calculated. As described earlier the tensor product of two states is quite a bit like a cartesian product with the added aspect of multiplication so let's try this out.

Consider the states $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ where $|0\rangle$ and $|1\rangle$ correspond to the eigenkets of a two state (ex: Spin-$\frac{1}{2}$) system. The cartesian product $(\times)$ would looks like this

$$|0\rangle \times |1\rangle = \begin{bmatrix} (1,0) \\ (1,1) \\ (0,0) \\ (0,1) \end{bmatrix}$$

and now multiplying the entries of the ordered pairs one gets what is known as the Kronecker Product which actualises the tensor product a matrix operation.

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} (1)(0) \\ (1)(1) \\ (0)(0) \\ (0)(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Great. We're now in good shape to try to describe a system comprised of two spin-$\frac{1}{2}$ particles. Let's call the state space of the first particle V_1 and of the second V_2 each with basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$. Tensor product space $V_1 \otimes V_2$ will have 4 basis vectors

$$|\uparrow_1\rangle \otimes |\uparrow_2\rangle \qquad |\uparrow_1\rangle \otimes |\downarrow_1\rangle \qquad |\uparrow_2\rangle \qquad |\downarrow_1\rangle \otimes |\downarrow_2\rangle$$

and considering the actual vector formulation allows us to calculate these new 4 dimensional basis vectors as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

giving

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

As such any state of the two particle system can be described as linear combination of these four vectors such that $|\psi\rangle \in V_1 \otimes V_2$ can be written as

$$|\psi\rangle = a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle + a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle + a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle$$

where $a_{ij} = \langle\psi| (|\uparrow_i\rangle \otimes |\downarrow_j\rangle) \in \mathbb{C}$ but we need to know how inner products work in tensor product spaces first, which gives me quite the neat segway to move forward.

Inner Products in Tensor Product Spaces The inner product of a bra and ket is of course as we know it, but in the tensor product of two state spaces $V \otimes W$ how is this going to work? Say I have

$$\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle$$

where we are taking the inner product of $|v_1 \otimes w_1\rangle$ and $|v_2 \otimes w_2\rangle$. Now much like the case when we were figuring out how operators work, each piece of this product will interact with the piece which lives in its own space giving

$$\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle = \langle v_1 | v_2 \rangle \langle w_1 | w_2 \rangle$$

where $\langle v_1 | v_2 \rangle$ is an inner product in V and $\langle w_1 | w_2 \rangle$ is an inner product in W.

Example We wish to find the normalised state in $V_1 \otimes V_2$ with zero total z and x-components, S_z^T and S_x^T of spin angular momentum. We will proceed by first finding the form of the state with zero total z-component and then find which the form of this state with zero x-component.

Our starting point is the generalised bipartite state

$$|\psi\rangle = a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle + a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle + a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle$$

To begin with, we remember that operators in tensor product spaces are upgraded and summed so as to enact on separate spaces in a way which which is mindful of both spaces. If the total z-component of angular momentum of a bipartite system is $S_{T_z} \in V_1 \otimes V_2$ and this can be represented as

$$S_{T_z} = S_{z_1} \otimes \mathcal{I} + \mathcal{I} \otimes S_{z_2}$$

Recall that S_z is characterised by the eigenket-eigenvalue relation $S_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$ and $S_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$. Beginning the calculation

$$S_{T_z}(|\psi\rangle) = (S_{z_1} \otimes \mathcal{I} + \mathcal{I} \otimes S_{z_2}) |\psi\rangle = (S_{z_1} \otimes \mathcal{I}) |\psi\rangle + (\mathcal{I} \otimes S_{z_2}) |\psi\rangle$$

and working out the first bit

$$(S_{z_1} \otimes \mathcal{I}) |\psi\rangle = (S_{z_1} \otimes \mathcal{I}) (a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle + a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle + a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle)$$
$$(S_{z_1} \otimes \mathcal{I}) |\psi\rangle = a_{11} S_{z_1} |\uparrow_1\rangle \otimes \mathcal{I} |\uparrow_2\rangle + a_{12} S_{z_1} |\uparrow_1\rangle \otimes \mathcal{I} |\downarrow_2\rangle + a_{21} S_{z_1} |\downarrow_1\rangle \otimes \mathcal{I} |\uparrow_2\rangle + a_{22} S_{z_1} |\downarrow_1\rangle \otimes \mathcal{I} |\downarrow_2\rangle$$
$$(S_{z_1} \otimes \mathcal{I}) |\psi\rangle = \frac{\hbar}{2} (a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle + a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle - a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle - a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle).$$

Similarly, the second bit will give us

$$(\mathcal{I} \otimes S_{z_2}) |\psi\rangle = \frac{\hbar}{2} (a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle - a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle - a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle).$$

Putting everything together we have

$$S_z^T |\psi\rangle = \hbar (a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle - a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle).$$

From this result is clear that S_z^T is 0 for $|\uparrow_1\rangle \otimes |\downarrow_2\rangle$ and $|\downarrow_1\rangle \otimes |\uparrow_2\rangle$ so now to find the state with zero S_x^T consider

$$|\psi\rangle = a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle$$

to which we enact S_x^T, again in pieces, recalling that $S_x |\uparrow\rangle = \frac{\hbar}{2} |\downarrow\rangle$ and $S_x |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$.

$$(S_{x_1} \otimes \mathcal{I}) |\psi\rangle = \left(S_x^1 \otimes \mathcal{I}\right) (a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle)$$
$$(S_{x_1} \otimes \mathcal{I}) |\psi\rangle = \frac{\hbar}{2} (a_{12} |\downarrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\uparrow_1\rangle \otimes |\uparrow_2\rangle)$$

Similarly,

$$(\mathcal{I} \otimes S_{x_2}) |\psi\rangle = \frac{\hbar}{2} (a_{21} |\downarrow_1\rangle \otimes |\downarrow_2\rangle + a_{12} |\uparrow_1\rangle \otimes |\uparrow_2\rangle)$$

so together we have

$$S_x^T |\psi\rangle = \frac{\hbar}{2} (a_{12} + a_{21}) (|\uparrow_1\rangle \otimes |\uparrow_2\rangle + |\downarrow_1\rangle \otimes |\downarrow_2\rangle) .$$

For this state to be have a 0 total x-component for angular momentum we thus require $a_{12} = -a_{21}$. Remembering that our earlier condition so that the z was 0 was $a_{11} = 0$ and $a_{22} = 0$ the state having both components equal to 0 must look like

$$|\psi\rangle = \alpha (|\uparrow_1\rangle \otimes |\downarrow_2\rangle - |\downarrow_1\rangle \otimes |\uparrow_2\rangle)$$

which is not normalised so let's make it so!

$$\langle\psi|\psi\rangle = \alpha^* \alpha (\langle\uparrow_1| \otimes \langle\downarrow_2| - \langle\downarrow_1| \otimes \langle\uparrow_2|) (|\uparrow_1\rangle \otimes |\downarrow_2\rangle - |\downarrow_1\rangle \otimes |\uparrow_2\rangle)$$
$$\langle\psi|\psi\rangle = |\alpha|^2 (\langle\uparrow_1 \mid \uparrow_1\rangle \otimes \langle\downarrow_2 \mid \downarrow_2\rangle - \langle\uparrow_1 \mid \downarrow_1\rangle \otimes \langle\downarrow_2 \mid \uparrow_2\rangle - \langle\downarrow_1 \mid \uparrow_2\rangle \otimes \langle\uparrow_2 \mid \downarrow_2\rangle + \langle\downarrow_1 \mid \downarrow_1\rangle \otimes \langle\uparrow_2 \mid \uparrow_2\rangle)$$
$$\langle\psi|\psi\rangle = |\alpha|^2 (1 - 0 - 0 + 1) \implies 2|\alpha|^2 = 1 \implies \alpha = \frac{1}{\sqrt{2}}.$$

This state,

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow_1\rangle \otimes |\downarrow_2\rangle - |\downarrow_1\rangle \otimes |\uparrow_2\rangle)$$

which we have shown to have total angular momentum 0 in the x and z directions and accepting on good faith that it has 0 in y also, has no total angular momentum and shall be our tool for inquiry in the coming section.

1.2 Describing Entangled States

When thinking of system with more than one particle we saw that the way to describe this situation would be by taking the tensor product of the state spaces of these respective particles. So for 2 particles we would have some $V \otimes W$. We also saw that the the basis vectors of this new space will be the tensor products of the basis vectors v_i and w_j of form $v_i \otimes w_j$. Indeed we qualified the fact that for two spin-$\frac{1}{2}$ systems, their tensor product state space will have in general states of the type

$$|\psi\rangle = a_{11} |\uparrow_1\rangle \otimes |\uparrow_2\rangle + a_{12} |\uparrow_1\rangle \otimes |\downarrow_2\rangle + a_{21} |\downarrow_1\rangle \otimes |\uparrow_2\rangle + a_{22} |\downarrow_1\rangle \otimes |\downarrow_2\rangle .$$

But, this doesn't mean that there can't be states $|\phi\rangle$ which can be described by one tensor product $v_k \otimes w_l \; \exists \, v_k \in V, \, w_l \in W$ which would mean that the first particle is in state v_k and the second is in w_l. That is, we can describe the particles independently! We can thus say that the two particles occupying state $|\phi\rangle$ are not entangled. Of course if there are no such v_k and v_l describing the multipartite state in terms of a single tensor product vector then we say that the particles occupying this state are entangled!

To clarify this idea, a non-entangled state is one where v_k is a linear comination

2 The Box & Coin Thought Experiment

3 The Density Operator

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[3] Witten, E. (2018). An Introduction to Information Theory.

[4] Zwiebach, B. (2013) Quantum Mechanics II Course Notes. Chapter 8. *Multiparticle States and Tensor Products*

[5] Preskill, J. (2015) Lecture Notes for Ph219/CS219: *Quantum Information*.