Notes on Quantum Optics

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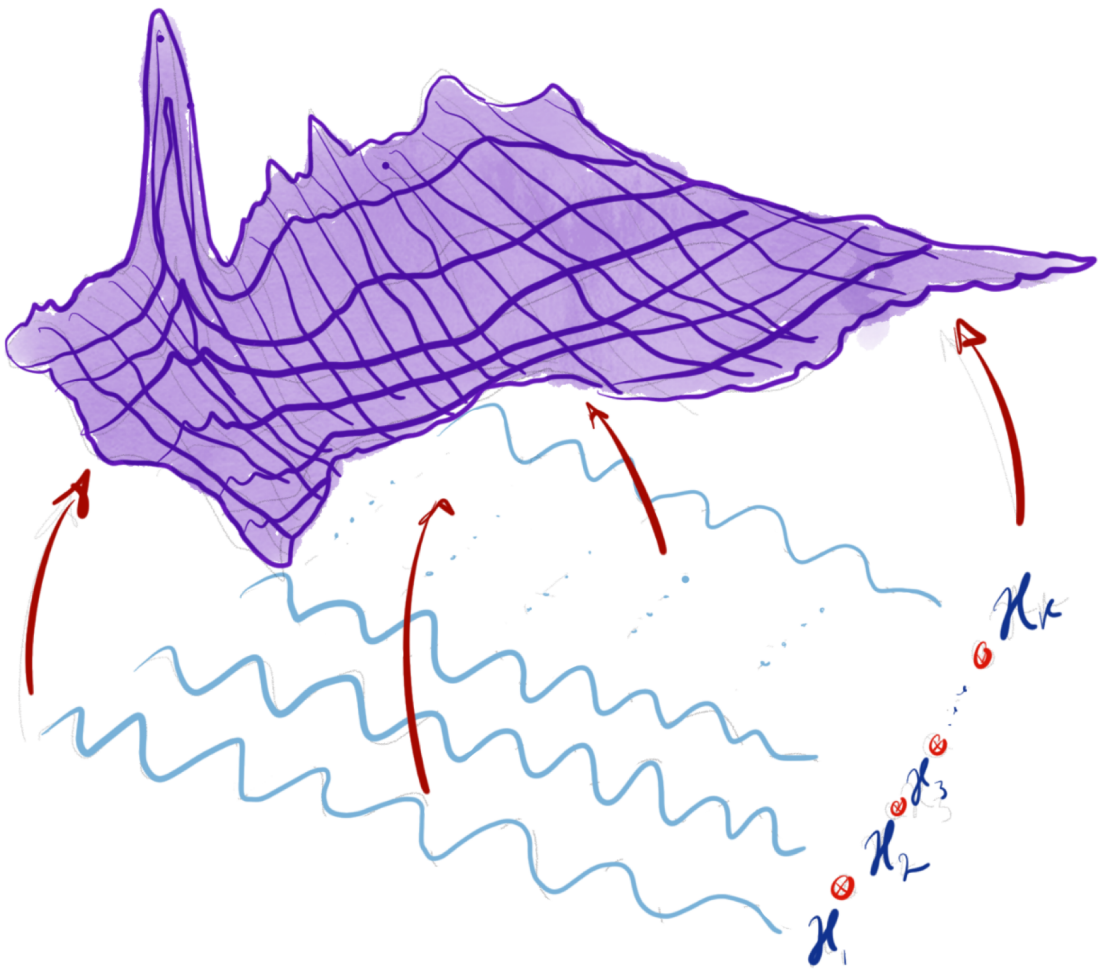


Hello reader! These are some notes I wrote on the material covered in PHY3227 - Quantum Optics, taught by Prof. André Xuereb at the University of Malta in Semester 1 of 2020-2021. I am writing these notes ahead of attempting the assignment for this unit and will be trying to cement my understanding of a number of concepts. They will be by no means a proper introduction to this subject.

Contents

1 Mathematical Preliminaries

1.1 Some Probability Theory

The use of statistics within physics and more so Quantum Mechanics is not something new. Indeed, density matrices emerge in this vain even within spaces of discrete variables but within the context of quantum mechanics with continuous variables, this mathematical language becomes even more prevalent.
Later on in these notes I will introduce the *Glauber - Sudarshan* Representation of states, which allows us to represent a density matrix in terms of the over-complete coherent basis as

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha.$$

Now, I'll talk about the physics later, but, this $P(\alpha)$ is a *quasi*probability distribution[1] over a continuous Hilbert space so let's talk about it as a mathematical object for a bit and introduce some ideas which will become very useful later on.

Classical Probability Distributions A probability distribution is a function from a sample space to the space of probabilities associated with the variables at hand. In our case we have a complex probability distribution

$$P : \mathcal{H} \to \mathbb{C}$$

[1]This pertains to a more generic class of functions known as generalised distributions including tempered and Schwartz distributions.

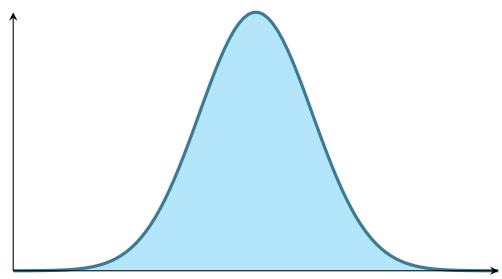
obeying the following three criteria known as the Kolmogorov Axioms.

1. Positivity $P(x) > 0 \, \forall \, x \in \mathcal{H}$
2. Unit Measure $\int P(x) \, dx = 1$
3. Additivity $P(A \cup B) = P(A) + P(B) \, : \, A \cap B = \emptyset$

For *quasi*probability distributions we see a violation of the 1st and 3rd Criteria in a way which is complimentary, in that we can have negative distributions which give can result in negative contributions to a sum within some intersection for example, reminiscent of wave interference.

Cementing this and moving towards cases we will interact with frequently, let's consider Gaussian distributions. These are in general, functions of which are an exponential of a quadratic polynomial such that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$



Here, we have μ the expectation value, σ the standard deviation or the square root of the variance and the normalisation factor $\frac{1}{\sqrt{2\pi}}$ given by the Gaussian integral which will be introduced in the coming lines. This is a probability density function for a Gaussian or Normal distribution so to find out the value of some random variable or continuous operator which follows such a distribution one would multiply by this density and integrate over the sample space.

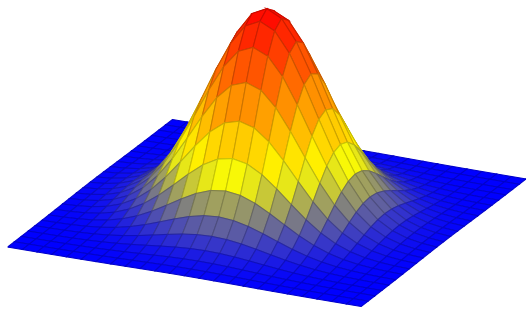
$$\langle x \rangle = \int x |f(x)|^2 dx$$

In quantum speak, recognise this as nothing other than finding the expectation value of the position of some state with wavefunction $|\psi\rangle = f(x)$

$$\langle x \rangle = \int x |f(x)|^2 dx = \int x \, \langle \psi | \psi \rangle \, dx = \int \langle \psi | \, x \, | \psi \rangle \, dx$$

and similarly the standard deviation can be expressed in terms of the variance as

$$\sigma_x \equiv \sqrt{\Delta x} \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle \psi | \, x^2 \, | \psi \rangle - \langle \psi | \, x \, | \psi \rangle^2}.$$



Covariance Matrices & Multivariable Gaussian Distributions In Quantum Mechanics, our Hilbert Spaces are built over the complex field[2] so our probability distributions[3] have to look like

$$P : \mathcal{H} \to \mathbb{C} \text{ or using an isomorphism } P : \mathcal{H} \to \mathbb{R}^2.$$

This means that we need to know at the very least how to speak about two variable Gaussian Distributions. Motivating this, consider again the 1D Gaussian function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

and turning x into a 2D vector we will have $\mathbf{x} = (x, y)$ and $\mu = (x_0, y_0)$ for some arbitrary expectation value but now the issue becomes how does one speak of the standard deviation? There are at least three different types of variances we can get here $\Delta(x)$, $\Delta(y)$, $\Delta(xy) = \Delta(yx)$[4] so might as well make a matrix out of them! This is known as the **covariance matrix**, which funnily enough is made out of variances and can be expressed for two variables as follows

$$\sigma = \begin{pmatrix} \Delta(x) & \Delta(xy) \\ \Delta(yx) & \Delta(y) \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle - \langle x \rangle^2 & \langle (xy)^2 \rangle - \langle xy \rangle^2 \\ \langle (yx)^2 \rangle - \langle yx \rangle^2 & \langle y^2 \rangle - \langle y \rangle^2 \end{pmatrix}.$$

Using this operator, we can now write out the two variable Gaussian as

$$f(\mathbf{x}) = \frac{1}{2\pi\sqrt{|\sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\mu)\sigma^{-1}(\mathbf{x}-\mu)^{\mathbf{T}}}$$

where $|\sigma|$ is the determinant of the covariance matrix and σ^{-1} is its inverse. This expression is generalisable up to arbitrary dimensions so for an n dimensional Gaussian one would write

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n |\sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\mu)\sigma^{-1}(\mathbf{x}-\mu)^{\mathbf{T}}}$$

where σ is an $n \times n$ positive semi-definite matrix and $\mathbf{x}$ and μ are n dimensional vectors.

[2] Physically motivating this, it's how we get wave behaviour.

[3] Wigner functions, are real probability distributions because they are distributions over phase space.

[4] Provided x and y commute.

The Gaussian Integral and Normalisation Consider the arbitrary Gaussian function $g(x) = e^{-x^2}$. How would one go about evaluating its integral over $(-\infty, \infty)$? Well, it's possible to show this directly by methods of analysis[5] but there's trick one can do to make this integral pliable using polar coordinates!
Consider the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

the issue here is evidently the evaluation at the bounds but consider that this is just a positive number so let's take its square root.

$$= \sqrt{\int_{-\infty}^{\infty} e^{-x^2}\, dx}\sqrt{\int_{-\infty}^{\infty} e^{-x^2}\, dx}$$

At this stage, x is just a dummy variable so let's change one of them to y, giving two single variable Gaussians

$$= \sqrt{\int_{-\infty}^{\infty} e^{-y^2}\, dx}\sqrt{\int_{-\infty}^{\infty} e^{-x^2}\, dx} = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy}$$

where I've used linearity of the integral operator and the distributivity of the square root over multiplication. Making use of polar coordinate we now are able to change the domain over which we are integrating from $x \in (-\infty, \infty)$ to $r \in (0, \infty)$, $\theta \in (0, 2\pi)$. This gives

$$\sqrt{\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\, drd\theta}$$

which is clearly a separable integral

$$\sqrt{\int_{0}^{2\pi} d\theta}\sqrt{\int_{0}^{\infty} e^{-r^2} r\, dr} = \sqrt{2\pi}\sqrt{\int_{0}^{\infty} e^{-r^2} r\, dr}.$$

Focusing on the unevaluated integral let's make a substitution of $z = -r^2$ giving

$$\sqrt{2\pi}\sqrt{\int_{-\infty}^{0} e^{-z} r\left(\frac{-1}{2r}\right) dz} = \sqrt{2\pi}\sqrt{\left[\frac{-e^{z}}{2}\right]_{-\infty}^{0}} = \sqrt{2\pi}\sqrt{\left[\frac{-e^{-r^2}}{2}\right]_{0}^{\infty}} = \sqrt{2\pi\left(\frac{1}{2}\right)}$$

and finally

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

[5] https://math.stackexchange.com/questions/2703724/gaussian-integral-using-single-integration

Now that we known what the area under a Gaussian function is we can normalise it to make it a probability distribution so consider the following Gaussian function with expectation value μ and variance σ^2

$$f(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$$

and let's compare it with a Gaussian function with $z = \frac{x-\mu}{\sqrt{2}\sigma}$, that is

$$f(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} = e^{-z^2}.$$

The integral of e^{-z^2} is clearly $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$, as has been shown, so making use of $z = \frac{x-\mu}{\sqrt{2}\sigma}$ we have by substitution

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} \left(\frac{1}{\sigma\sqrt{2}}\right) dx = \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

allowing us to write

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} dx = 1$$

deriving the normalisation factor.[6]

Moments The probability density is not the only way of describing a probability distribution. The moments of a distribution are the expectation values of the distribution for different powers of x. So the n-th moment would look like this

$$\mu_n = \langle x^n \rangle = \int_{-\infty}^{\infty} x^n f(x)\, dx.$$

These give information about the shape of the distribution and historically get their name from moments of inertia in mechanics where the first moment of a mass distribution is the center of mass and the second moment is the moment of inertia. Central moments are also used where the moment is shifted by the expectation value $\int_{-\infty}^{\infty} (x-\mu_1)^n f(x)\, dx$. Most notably, the variance is the second central moment

$$\begin{aligned} &\int_{-\infty}^{\infty} (x-\mu_1)^2 f(x)\, dx \\ = &\int_{-\infty}^{\infty} x^2 f(x) - 2x\mu_1 f(x) - {\mu^2}_1 f(x)\, dx \\ &= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 \\ &= \langle x^2 \rangle - \langle x \rangle^2 = \Delta x. \end{aligned}$$

These two moments are enough to uniquely determine Gaussian distributions[7] and this fact will allow us to at times do away completely with probability densities and speak only of covariance matrices, but more on this in pages to come.

[6] This proves $\int_{-\infty}^{\infty} e^{-\frac{z^2}{a}} dz = \sqrt{a\pi}$ as was asked in the appendix of the assignment.

[7] As it turns out, most well-known probability distributions may be described in such a way. Read more here.

For completeness, it is good to note the existence of *moment-generating* functions which are functions whose n-th derivative will correspond to the n-th moment of some distribution and as such they are alternate way of speaking about a distribution. The radius of convergence of a moment-generating function determines how many moments one needs to accurately speak of some distribution. This being said, the moment-generating function does not exist. When it does, it is the characteristic function, which always exists.

Characteristic Functions The Characteristic function can be thought of as the Fourier Transform of the probability density. To motivate this physically, consider that if I have a Gaussian distribution over position, then I will attain its characteristic function which will be a Gaussian function over momentum. It is even more remarkable that the derivatives of the characteristic function, give us the moments of the probability distribution readily. That is, it is a moment-generating function. For these reasons it is ubiquitous to study.

$$\varphi(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx = \langle e^{ipx} \rangle$$

Note that it is also expressible as a form of expectation value.

Operator Averages & Characteristics Functions for Operators The expectation value of an operator with respect to some distribution, which itself is an operator, can be expressed in terms of the trace operation in the following way in terms of say the Fock basis, which I'll introduce soon.

$$\begin{aligned}
\langle \mathbf{O} \rangle &= \sum_i \rho_i \langle i | \mathbf{O} | i \rangle \\
&= \sum_{i,j} \rho_i \langle i | \mathbf{O} | j \rangle \langle j | i \rangle \\
&= \sum_{i,j} \rho_i \langle j | i \rangle \langle i | \mathbf{O} | j \rangle \\
&= \sum_{i,j} \langle j | \rho_i | i \rangle \langle i | \mathbf{O} | j \rangle \\
\langle \mathbf{O} \rangle &= \mathrm{tr}(\rho \mathbf{O})
\end{aligned}$$

This is sometimes referred to as an operator average. Making use of this expression and the idea that the characteristic function can be written in terms of a type of expectation value we are in position to promote the characteristic function to operators over some distribution

$$\varphi(\xi) = \langle e^{i\mathbf{x}^{\mathbf{T}}\mathbf{\Omega}\xi} \rangle = \mathrm{tr}(\rho \mathbf{D}(\xi))$$

where I have defined the Weyl Operator $D(\xi) \equiv e^{i\mathbf{x}^{\mathbf{T}}\mathbf{\Omega}\xi}$ and made use of a symplectic form[8] which I will introduce now.

[8] One can see this as making sure the product of $\mathbf{x}$ and ξ respects the commutation relations.

1.2 Some results of Symplectic Geometry & Lie Algebra

Symplectic Forms & Transforms Quantum Mechanics in its phase space formulation, as we are discussing within this note set, makes use of the idea of canonical quantisation where Poisson brackets are promoted to commutators

$$\{f, g\} \to [\mathbf{f}, \mathbf{g}].$$

Now, Poisson brackets are *symplectic invariants* which is to say that they are artifacts of a larger structure, the same can be said of the commutator. These are both objects which satisfy the Jacobi Identity and form Lie Algebras but I will bite my tongue and not speak further on this topic. A symplectic form is a bilinear form with the quality that

$$\omega(x, y) = -\omega(y, x)$$

by definition both the Poisson bracket and the commutator make use of symplectic forms. Being a bilinear form this also defines another type of inner/scalar product in this space which leads to invariants. In particular, with operators we may make use of the symplectic matrix defined as follows

$$\Omega = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbb{I}_n \otimes \omega$$

So for a Hilbert Space $\mathcal{H}^{\otimes^n} = \bigotimes_{i=1}^{N} \mathcal{H}$ corresponding to N Bosonic modes which each have have a pair of annihilation and creation operators, we can form vectors $\mathbf{b} \equiv (\mathbf{a}_1, \mathbf{a}_1^\dagger, \mathbf{a}_2, \mathbf{a}_2^\dagger \ldots, \mathbf{a}_n, \mathbf{a}_n^\dagger)$ which allow us to write the Bosonic Commutation relations governing this space as

$$[\mathbf{b}_i, \mathbf{b}_j] = \mathbf{\Omega}_{i,j} \ : \ i, j \in [2N].$$

This also allows us to define Symplectic Transformations in the following way

$$\mathbf{S\Omega S^T} = \mathbf{\Omega} \implies S \in \mathsf{Sp}(2n, \mathcal{F})$$

where $\mathbf{S}$ can be called a symplectic matrix/transform/operator and is an element of the symplectic group over the field $\mathcal{F}$ of size $2n$, $\mathsf{Sp}(2n, \mathcal{F})$. The symplectic matrix is clearly itself a symplectic transform.

Williamson's Theorem An operator M is said to be *positive-definite* operator if and only if all of its eigenvalues are positive. Take my word for it, symplectic matrices are able to diagonalise positive-definite matrices.

Theorem: Let M be a real positive-definite symmetric real $2n \times 2n$ matrix.

There exists $\mathbf{S} \in \mathsf{Sp}(2n, \mathcal{F})$ such that:

$$\mathbf{S^T M S} = \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix}$$

with ν being a diagonal matrix of rank n. $\nu = \mathsf{diag}(\nu_1, \ldots, \nu_n)$ is known as the symplectic spectrum of M and involves the symplect eigenvalues of M which are invariant up to reordering under symplectic transformation. As such and making use of the fact that $\mathbf{\Omega} = \mathbf{\Omega}^T = \mathbf{\Omega}^{-1} \in \mathsf{Sp}(2n, \mathcal{F})$ we can find the symplectic spectrum directly as follows

$$\nu = \mathsf{Eig}_+(i\mathbf{\Omega M}).$$

Baker-Campbell-Haussdorf Formula In a non-commutative algebra, it is not easy to find an operator $\mathbf{Z}$ which satisfies the following relationship given $\mathbf{X}$ and $\mathbf{Y}$

$$e^{\mathbf{X}}e^{\mathbf{Y}} = e^{\mathbf{Z}}.$$

The Baker-Campbell-Hausdorff formula is the solution to this question giving us the following series

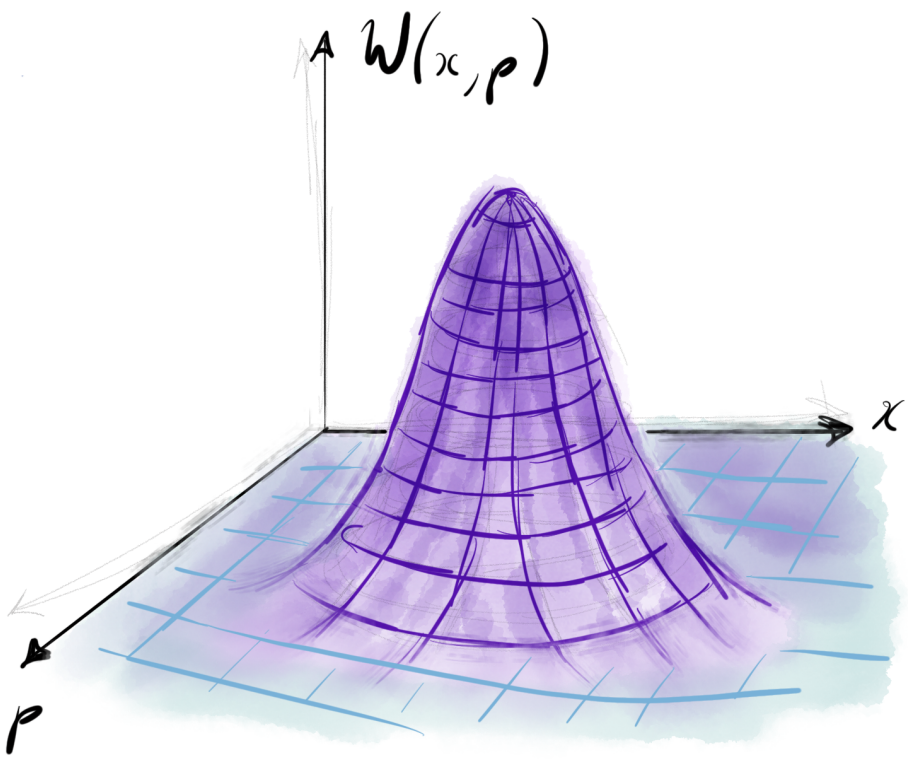
$$\mathbf{Z} = \mathbf{X} + \mathbf{Y} + +\mathbf{Z} + \frac{1}{12}[\mathbf{X}, [\mathbf{X}, \mathbf{Y}]] - \frac{1}{12}[\mathbf{Y}, [\mathbf{X}, \mathbf{Y}]] \dots$$

which in some sense gives us an idea of how non-commutative this algebra is. In Quantum Mechanics, as a result of the canonical bosonic commutation relations our operators form part of the Heisenberg group, which has a commutativity that reduces this formula in the following way

$$e^{\mathbf{X}}e^{\mathbf{Y}} = e^{\mathbf{X}+\mathbf{Y}+\frac{1}{2}[\mathbf{X},\mathbf{Y}]}.$$

2 Quantum Mechanics with Continuous-Variables

Having dealt with the majority of the mathematics needed, I will now introduce the physics of the quantised electromagnetic field, photons. In particular, we will deal with a quantum state space where the operators are continuous, like position and momentum, and their phase spaces with probability distributions over them. This will represent an approach which is not only physical but is useful in an information sense, leading to an understanding of many specifications of quantum states which are not readily obvious in other approaches. In particular, we will focus on Gaussian States since they are so pliable, nice to deal with and as is frequently done in Quantum Optics.



Density Matrix in Phase Space

$$\rho \longrightarrow \chi(\eta) = \mathrm{Tr}\{\rho e^{\eta \hat{a}^\dagger - \eta^* \hat{a}}\} \longrightarrow W(\alpha) = \frac{1}{\pi^2}\int e^{\alpha \eta^* - \alpha^* \eta}\chi(\eta)\, d^2\eta$$

Density Matrix Characteristic Function Wigner Function

2.1 Quantisation & Bosonic Field Operators

A lot of Quantum Optics textbooks start with a brief chapter *quantising* the electromagnetic field within a cavity but I think this is quite cheap. If you'd like a proper review of the canonical quantisation of the

electromagnetic field I recommend David Tong's Notes on QFT. I will instead motivate the annihilation and creation operators in a way which is more instructive via the harmonic oscillator and quantum harmonic oscillator which will serve as more than enough field theory for us to move on to quantum optics with Gaussian States.

The Harmonic Oscillator Sidney Coleman once remarked that *"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."*. Harmonic Oscillators have the following equation of motion[9]

$$\ddot{x} + \omega_0^2 x = 0$$

where x is the coordinate of the oscillator and ω_0 is the natural resonance frequency. This gives a Lagrangian of form

$$\mathcal{L} = \frac{\beta}{2}\dot{x}^2 - \frac{1}{2\alpha}x^2\,,$$

then applying the Euler-Lagrange equation gives

$$\begin{aligned}\frac{d\mathcal{L}}{dx} - \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{x}} &= 0\\ \ddot{x} + \frac{1}{\alpha\beta}x &= 0\,,\end{aligned}$$

which is correct for any α and β such that $1/\alpha\beta = \omega_0^2$.

Following the usual procedure to find the Hamiltonian, we get

$$H = \frac{1}{2\alpha}x^2 + \frac{1}{2\beta}p^2$$

where the momentum p is defined as $p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} = \beta\dot{x}$. Hamilton's equations of motion are

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{-x}{\alpha} \qquad \text{and} \qquad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\beta}$$

or combined as a matrix equation

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{pmatrix} 0 & 1/\beta \\ -1/\alpha & 0 \end{pmatrix}\begin{bmatrix} x \\ p \end{bmatrix}.$$

If we define $x \equiv Ax$ and $y \equiv Bp$ with the constraint that $A/B = \sqrt{\beta/\alpha}$, then our Hamilton equations of motion become

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \omega_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

[9] Reference was made to https://physics.stackexchange.com/questions/432823/how-to-derive-annihilation-and-creation-operators

This is a set of first order coupled differential equations for x and y. To uncouple the equations, we solve for the eigenvectors and eigenvalues of the matrix. They are

$$a \equiv x + iy = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ with eigenvalue } i\omega$$

and

$$a^* \equiv x - iy = \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ with eigenvalue } -i\omega\,.$$

Note that we have just found the eigenvalues of a symplectic matrix. These already look like the annihilation and creation operators, but let's make this relationship more explicit.

The Quantum Harmonic Oscillator Consider again the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

with Poisson bracket $\{x, p\} = 1$ and promote this to a commutator $[\hat{X}, \hat{P}] = i\hbar$, respecting the uncertainty relations, where $\hat{P} = \frac{p}{\sqrt{m}}$ and $\hat{X} = \sqrt{m}\omega x = \sqrt{k}$ giving

$$\hat{H} = \hat{P}^2 + \hat{X}^2$$

where now, inspired by the expressions which diagonalised the classical hamiltonian one writes

$$\hat{H} = \frac{\left(\hat{X} + i\hat{P}\right)}{\sqrt{2}} \frac{\left(\hat{X} - i\hat{P}\right)}{\sqrt{2}} + \frac{i}{2}\sqrt{\frac{k}{m}}[\hat{P}, \hat{X}]$$
$$\hat{H} = \frac{\left(\hat{X} + i\hat{P}\right)}{\sqrt{2}} \frac{\left(\hat{X} - i\hat{P}\right)}{\sqrt{2}} + \frac{\hbar\omega}{2}$$

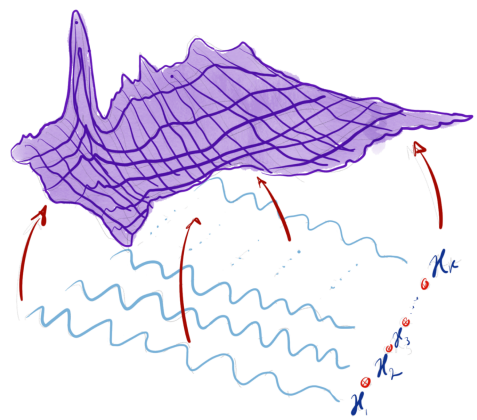
and so definition of a hermitian conjugate follows

$$\hat{a} \equiv \frac{\left(\hat{X} - i\hat{P}\right)}{\sqrt{2\hbar\omega}} \qquad \hat{a}^\dagger \equiv \frac{\left(\hat{X} + i\hat{P}\right)}{\sqrt{2\hbar\omega}}$$

resulting in the Bosonic Commutation Relation $[\hat{a}, \hat{a}^\dagger] = 1$ and giving the diagonalised Hamiltonian as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{\mathbf{1}}{2}\right).$$

These operators are known as the annihilation and creation operators respectively and the eigenspectrum of their product, the number operator, describes the energy levels of this quantum harmonic oscillator.



A State Space for the Quantised Electromagnetic Field The modes of the electromagnetic field can be written out in terms of our new friends the annihilation and creation operators. Making use of the Coulomb Gauge one arrives to the vector potential

$$\mathbf{A}(\mathbf{r},t) = \sum_k \left(\frac{\hbar}{2\omega_k \epsilon_0}\right)^{\frac{1}{2}} \left[a_k \mathbf{u_k}(\mathbf{r}) e^{-i\omega_k t} + a_k^\dagger \mathbf{u_k}^*(\mathbf{r}) e^{i\omega_k t}\right]$$

which is related to the magnetic and electric field in the following ways

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad\qquad \mathbf{E} = \frac{-\partial \mathbf{A}}{\partial t}.$$

The Hamiltonian of the electromagnetic field is expressed as

$$H = \frac{1}{2}\int \epsilon_0 \mathbf{E^2} + \frac{1}{\mu_0}\mathbf{B^2} d\mathbf{r}$$

and expressing this Hamiltonian in terms of the gauge potential leads to

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a_k}^\dagger \hat{a_k} + \frac{\mathbf{1}}{2}\right)$$

where we recover the Hamiltonian of the quantum harmonic oscillator, indeed we have k of them each representing a mode of the electromagnetic field. Using the Hamiltonian, one can derive an eigenbasis for this space and we will do this shortly. Before this a mathematical look at this space shows a complex inner product space, a Hilbert Space, of infinite dimension since the Hamiltonian offers a continuous spectrum.

If we have multiple such systems, say two photons, then we take the tensor product of the underlying spaces to speak of them together. This means that for an n-dimensional quantum system we have h

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$$

where each of the composite spaces, is an infinite dimensional Hilbert space in its own right, armed with its own pair of creation and annihilation operators which observe the canonical Bosonic commutation relation

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}.$$

This space is also armed with a non-trivial symplectic form (stemming from the promotion of the Poisson brackets to commutators).
Making use of the earlier definitions of the annihilation and creation operators we can express the quadrature phase operators, or *position and momentum*[10] of each of these quantum harmonic oscillators in the following way reversing the earlier derivation of these operators and setting dimensionless constants to 1.

$$\hat{x}_i = \frac{\hat{a}_i + \hat{a}_i^\dagger}{\sqrt{2}} \qquad \hat{p}_i = \frac{\hat{a}_i + \hat{a}_i^\dagger}{i\sqrt{2}}$$

So as to speak of the quadratures of the whole space, the vector of operators $\mathbf{R} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_n, \hat{p}_n)$ is introduced allowing us to write the canonical commutation relation observed by this space

$$[\mathbf{R}_k, \mathbf{R}_l] = i\mathbf{\Omega}$$

where $\mathbf{\Omega}$ is the symplectic matrix introduced prior. And that's ... *all* the maths you need to live here.

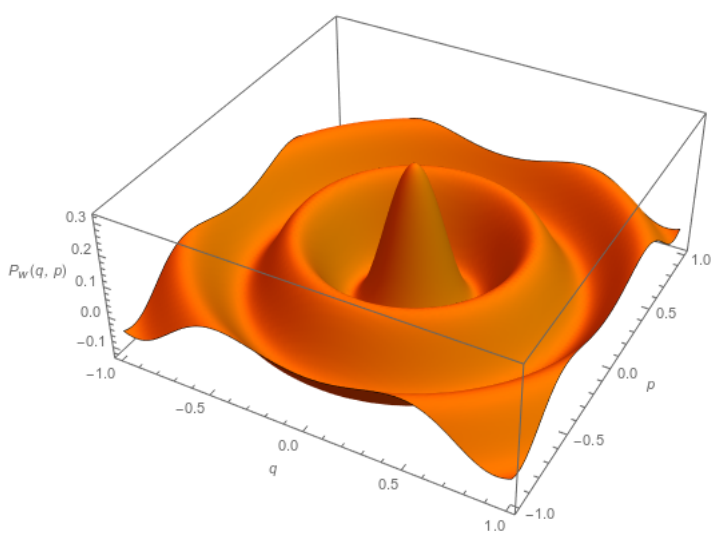


Figure 1: $|18\rangle$ in all its glory.

[10] This is really and truly a dangerous adoption of names from classical phase space. These are not the position and momentum of the electromagnetic field but rather must be thought of as the real and imaginary parts of the complex amplitude of $\hat{a}$.

2.2 A Zoo of States in Phase Space

Fock States By the Spectral Theorem we know that the eigenvectors of $\hat{H}$ provide a basis for the space. Clearly the eigenvectors of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ are the eigenvectors of $\hat{H}$ and so provide a basis for the Hilbert space of a mode. These states will satisfy the following relations with the number, creation and annihilation operators.

$$\hat{n}\,|n\rangle = \hat{a}^\dagger \hat{a}\,|n\rangle = n\,|n\rangle \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

Since they form a basis in Hilbert Space, they are orthogonal and satisfy completeness

$$\langle n|m\rangle = \delta_{n,m} \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}.$$

For emphasis, say I have a 2D quantum state living in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the basis states of this space would look like $|n\rangle_1 \otimes |m\rangle_2$. Physically, these states correspond directly to the energy levels of this quantum harmonic oscillator which is modeling our mode and conceptually are said to represent the number of photons occupying a mode, although I have my own philosophical qualms with this. We'll examine the mathematical form of these states in just a bit when we talk about characteristic functions.

Later we will compare the uncertainties in the quadratures of different states. Let's find the uncertainty relations for position and momentum for Fock states. Some things to note here are that $\Delta\hat{x} = \Delta\hat{p}$ and

$$\Delta\hat{x} = \left(\langle n|\hat{x}^2|n\rangle - \langle n|\hat{x}|n\rangle^2\right)^{1/2}$$

$$\langle n|\hat{x}|n\rangle = \langle n|\frac{(\hat{a}+\hat{a}^\dagger)}{\sqrt{2}}|n\rangle = \frac{1}{\sqrt{2}}\left[\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle\right] = \frac{1}{\sqrt{2}}\left[\sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle\right] = 0$$

$$\langle n|\hat{x}^2|n\rangle = \frac{1}{2}\langle n|(\hat{a}+\hat{a}^\dagger)^2|n\rangle = \frac{1}{2}\langle n|\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}|n\rangle$$

$$= \frac{1}{2}\langle n|\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+2\hat{a}^\dagger\hat{a}|n\rangle$$

$$= \frac{1}{2}\left[\langle n|\hat{a}^2|n\rangle + \langle n|\hat{a}^{\dagger 2}|n\rangle + 1 + 2\langle n|\hat{a}^\dagger\hat{a}|n\rangle\right]$$

$$= \frac{1}{2}\left[0+0+1+2\sqrt{n}\,\langle n|\hat{a}^\dagger|n-1\rangle\right] = \frac{1}{2}\left[1+2\sqrt{n}\sqrt{n}\,\langle n|n\rangle\right]$$

$$= n+\frac{1}{2}$$

Similarly, $\langle n|\hat{p}^2|n\rangle = \frac{-1}{2}\langle n|(\hat{a}-\hat{a}^\dagger)^2|n\rangle = \frac{-1}{2}\langle n|\hat{a}^2+\hat{a}^{\dagger 2}-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}|n\rangle$

$$= \frac{-1}{2}\left(-\langle n|\hat{a}\hat{a}^\dagger|n\rangle - \langle n|\hat{a}^\dagger\hat{a}|n\rangle\right) = \frac{-1}{2}\left(-(n+1)-n\right)$$

$$= n+\frac{1}{2}$$

$$\therefore \quad \Delta\hat{x}\,\Delta p = \sqrt{n+\frac{1}{2}}\sqrt{n+\frac{1}{2}} = n+\frac{1}{2} \geq \frac{1}{2}$$

that the Heisenberg Uncertainty relation is observed for $\hbar = 1$.

Displacement & Coherent States The complex form of the Weyl operator which was introduced earlier is the displacement operator. Conceptually, it is able to translate or displace states within the optical phase space whilst mathematically it allows us to generate the eigenstates of the annihilation operator, known as the coherent states.

$$\hat{D}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \ : \ \alpha \in \mathbb{C}$$

Making use of the Baker-Campbell-Haussdorf Formula, the Displacement operator can be written alternatively as follows

$$\hat{D}(\alpha) = e^{-\frac{-|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}.$$

The Displacement operator lives up to its name affecting the creation and annihilation operators in the following way

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha \qquad\qquad \hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*.$$

Its most notable characteristic is that it generates coherent states from the vacuum state. The coherent states are eigenstates of the annihilation operator.

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle \qquad\qquad \hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,.$$

Using all of this information we can now derive the uncertainty relations for coherent states. Evidently,

$$\langle\hat{x}\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \langle\alpha|\frac{(\hat{a}+\hat{a}^\dagger)}{\sqrt{2}}|\alpha\rangle = \frac{1}{\sqrt{2}}\left[\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right]$$

$$\langle\alpha|\hat{a}|\alpha\rangle = \langle 0|\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)|0\rangle = \langle 0|(\hat{a}+\alpha)|0\rangle = \langle 0|\hat{a}|0\rangle + \alpha\langle 0|0\rangle = \alpha$$

$$\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \langle 0|\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)|0\rangle = \langle 0|(\hat{a}^\dagger+\alpha^*)|0\rangle = \langle 0|\hat{a}^\dagger|0\rangle + \alpha^*\langle 0|0\rangle = \alpha^*$$

$$\Longrightarrow \langle\hat{x}\rangle = \frac{\alpha+\alpha^*}{\sqrt{2}}$$

$$\langle\hat{x}^2\rangle = \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{1}{2}\langle\alpha|(\hat{a}+\hat{a}^\dagger)^2|\alpha\rangle = \frac{1}{2}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}|\alpha\rangle$$

$$= \frac{1}{2}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+2\hat{a}^\dagger\hat{a}|\alpha\rangle] = \frac{1}{2}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1|\alpha\rangle$$

$$\langle\alpha|\hat{a}^2|\alpha\rangle = \langle\alpha|\hat{a}\hat{a}|\alpha\rangle = \alpha\langle\alpha|\hat{a}|\alpha\rangle = \alpha^2$$

$$\langle\alpha|\hat{a}^{\dagger 2}|\alpha\rangle = \langle 0|\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}^\dagger(\alpha)|0\rangle = \langle 0|(\alpha^*+\hat{a}^\dagger)^2|0\rangle$$

$$= \langle 0|\hat{a}^{\dagger 2}+2\alpha^*\hat{a}^\dagger+\alpha^{*2}|0\rangle = \langle 0|\hat{a}^{\dagger 2}|0\rangle + 2\alpha^*\langle 0|\hat{a}^\dagger|0\rangle + \alpha^{*2}$$

$$= \alpha^{*2}$$

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha\langle 0|\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)|0\rangle = \alpha\langle 0|(\alpha^*+\hat{a}^\dagger)|0\rangle = \alpha\alpha^* + \langle 0|\hat{a}^\dagger|0\rangle$$

$$= \alpha\alpha^* = |\alpha|^2$$

$$\Longrightarrow \langle\hat{x}^2\rangle = \frac{\alpha^2+\alpha^{*2}+2|\alpha|^2+1}{2} = \frac{(\alpha+\alpha^*)^2+1}{2}$$

similarly,

$$\langle\hat{p}\rangle = \langle\alpha|\hat{p}|\alpha\rangle = \langle\alpha|\frac{(\hat{a}-\hat{a}^\dagger)}{\sqrt{2}i}|\alpha\rangle = \frac{i}{\sqrt{2}}\left[\langle\alpha|\hat{a}|\alpha\rangle - \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right]$$

$$\Longrightarrow \langle\hat{p}\rangle = \frac{i(\alpha-\alpha^*)}{\sqrt{2}}$$

similarly,

$$\langle\hat{p}^2\rangle = \langle\alpha|\left(\frac{i(\hat{a}-\hat{a}^\dagger)}{\sqrt{2}}\right)^2|\alpha\rangle = \frac{-1}{2}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger 2}-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}|\alpha\rangle$$

$$= \frac{-1}{2}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger 2}-\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}-2\hat{a}^\dagger\hat{a}|\alpha\rangle = \frac{-1}{2}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger 2}-2\hat{a}^\dagger\hat{a}-1|\alpha\rangle$$

$$\Longrightarrow \langle\hat{p}^2\rangle = -\frac{1}{2}\left[\alpha^2+\alpha^{*2}-2|\alpha|^2-1\right] = \frac{(\alpha-\alpha^*)^2-1}{2}$$

$$\Delta\hat{x} = \left[\langle\alpha|\hat{x}^2|\alpha\rangle - \langle\alpha|\hat{x}|\alpha\rangle^2\right]^{1/2} = \left[\frac{(\alpha+\alpha^*)^2+1-(\alpha+\alpha^*)^2}{2}\right]^{1/2}$$

$$= \frac{1}{\sqrt{2}}$$

$$\Delta\hat{p} = \left[\langle\alpha|\hat{p}^2|\alpha\rangle - \langle\alpha|\hat{p}|\alpha\rangle^2\right]^{1/2} = \left[\frac{(\alpha-\alpha^*)^2-1-(\alpha-\alpha^*)^2}{2}\right]^{1/2}$$

$$= 1/\sqrt{2}$$

$$\therefore \ \Delta\hat{x}\,\Delta\hat{p} = 1/2$$

these states have an uncertainty which is the minimum possible and are in this vain as close to classical as we can get within quantum optics. Another way to understand this is that these states are translated vacuum states. The states being the spectrum of an operator of this space, the annihilation operator, form a basis form this space and thus satisfy completeness to the point that they form an overcomplete basis.

$$\mathbf{1} = \frac{1}{2\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha$$

An inner product of a Fock state and a coherent state can be shown to observe

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{\frac{1}{2}}}\langle 0|\alpha\rangle \text{ where } \langle 0|\alpha\rangle = \langle 0|\,\hat{D}(\alpha)\,|0\rangle = e^{\frac{-|\alpha|^2}{2}}$$

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{\frac{1}{2}}}e^{\frac{-|\alpha|^2}{2}}$$

allowing us to write a direct relationship between the Fock basis and the Coherent basis as follows

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\,|\alpha\rangle = \sum_n |n\rangle\,\langle n|\alpha\rangle = \sum_n |n\rangle\,\frac{\alpha^n}{(n!)^{\frac{1}{2}}}e^{\frac{-|\alpha|^2}{2}}.$$

The inner product between two coherent states can also be written explicitly using the Displacement operator as

$$\langle\beta|\alpha\rangle = \langle 0|\,\hat{D}^\dagger(\beta)\hat{D}(\alpha)\,|0\rangle = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha\beta^*}$$

giving

$$|\,\langle\beta|\alpha\rangle\,|^2 = e^{-|\alpha-\beta|^2}$$

showing that the states become approximately orthogonal in the limit $|\alpha-\beta| >> 1$ and explaining the overcompleteness of the basis.

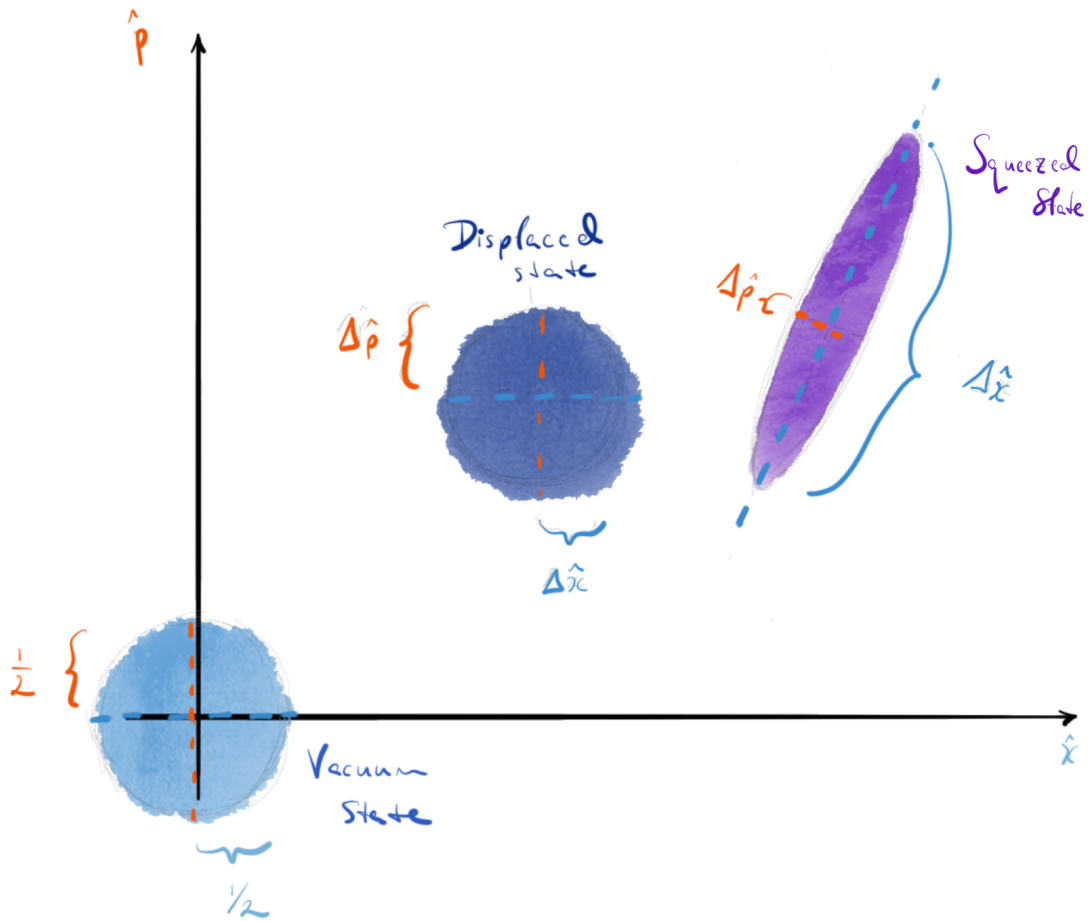
Oh and finally and because I don't want to mess up my figure placement, the displacement operator is a Unitary operator

$$\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$$

and using the Baker-Campbell-Haussdorf Formula we can write products of Displacement operators in the following way

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\frac{1}{2}(\beta^*\alpha-\alpha^*\beta)}\hat{D}(\alpha+\beta).$$

Phase Space Just as in classical mechanics the notion of a phase space arises to encapsulate all the possible states a system can occupy we can do this in quantum optics too. In particular, we are able to visualise the uncertainty associated with an expectation value of some state. The Fock states and Coherent states both clearly form circles in phase space as they have $\Delta\hat{x} = \Delta\hat{p}$. Soon we will see that squeezed states form ellipses and that we can visualise and describe density matrices completely as distributions over phase space using characteristic functions. This picture also allows us to clearly see the effect of the



displacement operator as the phase plot of a coherent state is clearly a displaced version of the phase plot of the vacuum state, as is evidenced by their uncertainty relations.

Squeezing in Phase Space Inspired by this picture, we could try to construct a minimum uncertainty state, a state with the same uncertainty as a coherent state, but one where the quadratures do not have equal uncertainty. Such states are known as squeezed states and are a general class of minimum uncertainty states which the coherent states can thought to be a specific class of squeezed states.

The Squeezing operator is defined as follows

$$\hat{S}(\epsilon) = e^{\frac{1}{2}\epsilon^*\hat{a}^2 - \frac{1}{2}\epsilon\hat{a}^{\dagger 2}} \ : \ \epsilon = re^{i2\phi}$$

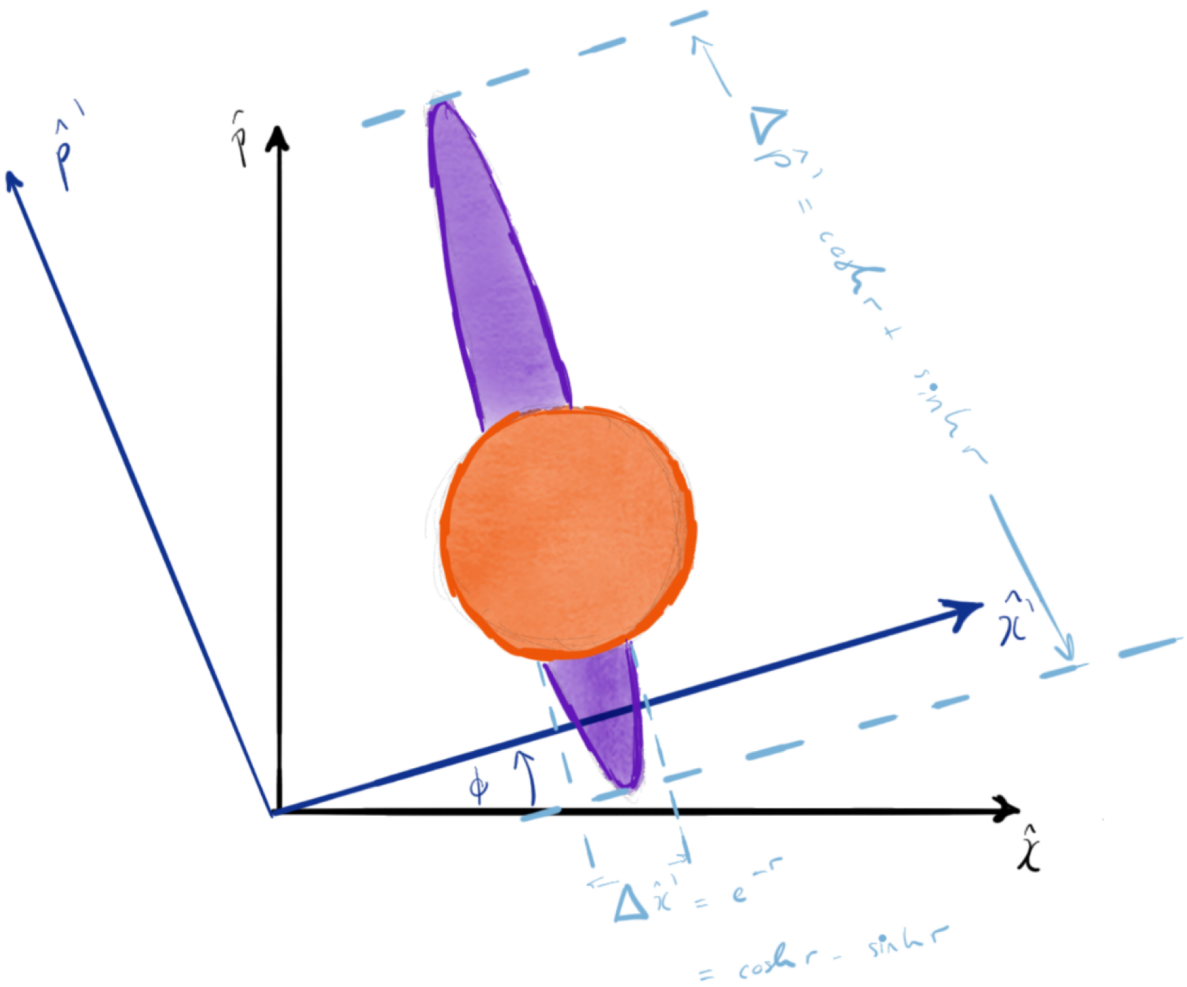
and transforms the annihilation and creation operators in the following way

$$\hat{S}^\dagger \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \hat{S} = \begin{pmatrix} \hat{a}\cosh r - \hat{a}^\dagger e^{i2\phi}\sinh r \\ \hat{a}^\dagger \cosh r - \hat{a}e^{-i2\phi}\sinh r \end{pmatrix}.$$

It is a unitary operator satisfying the following relations

$$\hat{S}^\dagger(\epsilon) = \hat{S}^{-1}(\epsilon) = \hat{S}(-\epsilon).$$

The effect of the squeezing operator on the two quadratures can be seen to be a rotation by $e^{-i\phi}$ so



defining

$$\hat{x}' + i\hat{p}' = (\hat{x} + i\hat{p})\, e^{-i\phi}$$

one is able to calculate[11] the expectation values for the rotated quadratures as $\Delta\hat{x}' = \frac{1}{\sqrt{2}}e^{-r}$ and $\Delta\hat{p}' = \frac{1}{\sqrt{2}}e^{r}$ allowing us to recover the minimum uncertainty relation

$$\Delta\hat{x}'\Delta\hat{p}' = \frac{1}{2}.$$

Not that in this case $\Delta\hat{x}' \neq \Delta\hat{p}'$. In fact we have the position quadrature being attenuated and the momentum quadrature being amplified (the example in the figure is reversed) giving us an ellipse in phase space, a *squeezed* circle.

2.3 Charateristic Functions for Phase Space

We have seen that coherent states are quantum states which constitute a minimum uncertainty basis for quantum states of bosonic continuous variable quantum systems and that they form nice circles of radius $\frac{1}{2}$ in phase space, but these are not general states which we meet normally. Indeed, states can be formed as statistical ensembles of other states. Our prior education in quantum mechanics has armed us with a tool known as density matrices for just a scenario but in a continuous variable system we can't some over outer products we have to integrate

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha.$$

This $P(\alpha)$ is an unwieldy and tough density function to deal with directly. In such a case, it proves helpful to take a page out of statistics and take a look at the characteristic function of this distribution and from it attain the density function indirectly, since these quantities are related by a Fourier Transform.

The Characteristic function is expressed in terms of the Weyl Operator as

$$\chi(\xi) = \mathbf{tr}(\rho D(\xi)) = \mathbf{tr}(\rho e^{i\mathbf{x}^T\mathbf{\Omega}\xi})$$

where we have a function over $\xi \in \mathbb{R}^{2N}$. Let's make this more explicit for a single mode state with phase quadratures $\mathbf{x}^T = (\hat{x}, \hat{p})$ where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and we can take $\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}$.

$$i\mathbf{x}^T\mathbf{\Omega}\xi = i\left(\beta\hat{x} - \alpha\hat{p}\right)$$

expanding in terms of annihilation and creation operators

$$= \frac{1}{\sqrt{2}}\left(-\alpha\hat{a} + i\beta\hat{a} + \alpha\hat{a}^\dagger + i\beta\hat{a}^\dagger\right)$$

where if we define $\eta = \frac{\alpha + i\beta}{\sqrt{2}}$ we get

$$i\mathbf{x}^T\mathbf{\Omega}\xi = \eta\hat{a}^\dagger - \eta^*\hat{a}.$$

which is what you'll find in Walls & Milburn. As such, the characteristic function can be written as

$$\chi(\eta) = \mathbf{tr}(\rho e^{\eta\hat{a}^\dagger - \eta^*\hat{a}}).$$

[11] These calculations are given in reasonable depth on pg.65 of Quantum Optics by Scully & Zubairy.

Now, looking at this characteristic function we see that we can have three different operator orderings symmetric, normal and anti-normal ordering

$$\chi_S(\eta) = \mathbf{tr}(\rho e^{\eta\hat{a}^\dagger - \eta^*\hat{a}}) \qquad \chi_N(\eta) = \mathbf{tr}(\rho e^{\eta\hat{a}^\dagger} e^{-\eta\hat{a}^*}) \qquad \chi_A(\eta) = \mathbf{tr}(\rho e^{-\eta^*\hat{a}} e^{\eta\hat{a}^\dagger})$$

giving three different density functions through an inverse Fourier transform.

$$W(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi_S(\eta)\, d^2\eta \quad P(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi_N(\eta)\, d^2\eta \quad Q(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi_A(\eta)\, d^2\eta$$

Respectively known as, the Wigner Function, the Glauber Sudarshan P-Function and the Husimi Q-Function. For this note set we'll focus on Wigner functions.// Let's do some examples!

The Wigner Function for a Coherent State The thing to keep in mind here is that since the characteristic function and the fourier transform term run over the same domain in the integral they must be evaluated together. Aside from this, complex numbers are split into their real and imaginary parts for evaluation of the integral and a standard integral is used.

$$W(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi_s(\eta)\, d^2\eta$$

so for $\rho = |\alpha_0\rangle\langle\alpha_0|$ we have

$$= \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\, d^2\eta$$

$$= \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\langle 0|\hat{D}^\dagger(\alpha_0)\hat{D}(\eta)\hat{D}(\alpha_0)|0\rangle\, d^2\eta$$

$$= \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*} e^{\frac{i}{2}(\alpha_0^*\eta - \eta^*\alpha_0)}\langle 0|\hat{D}(-\alpha_0)\hat{D}(\alpha_0+\eta)|0\rangle\, d^2\eta$$

$$= \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*} e^{\frac{i}{2}(\alpha_0^*\eta - \eta^*\alpha_0)} e^{\frac{i}{2}(\alpha_0+\eta)^*(-\alpha_0) - \frac{i}{2}(-\alpha_0)^*(\alpha_0+\eta)}\langle 0|\hat{D}(\eta)|0\rangle\, d^2\eta$$

$$= \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*} e^{\frac{i}{2}(\alpha_0^*\eta - \eta^*\alpha_0)} e^{\frac{i}{2}(\cancel{\alpha_0\alpha_0^*} - \alpha_0\eta^* + \cancel{\alpha_0^*\alpha_0} + \alpha_0^*\eta)} e^{-|\eta|^2/2}\, d^2\eta$$

$$= \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^* + \alpha_0^*\eta - \alpha_0\eta^*} e^{-|\eta|^2/2}\, d^2\eta$$

$$= \frac{1}{\pi^2}\int e^{\eta^*(\alpha-\alpha_0)} e^{-\eta(\alpha-\alpha_0)^*} e^{-|\eta|^2/2}\, d^2\eta$$

Let $\eta = \eta_r + i\eta_i$ & $\alpha - \alpha_0 = (\alpha_r - \alpha_{0r}) + i(\alpha_i - \alpha_{0i})$

$$\Rightarrow \frac{1}{\pi^2}\int e^{(\eta_r - i\eta_i)[(\alpha_r-\alpha_{0r}) + i(\alpha_i-\alpha_{0i})] - (\eta_r + i\eta_i)[(\alpha_r-\alpha_{0r}) - i(\alpha_i-\alpha_{0i})] - \frac{1}{2}(\eta_r^2+\eta_i^2)}\, d\eta_r\, d\eta_i$$

$$= \frac{1}{\pi^2}\int e^{2i\eta_r(\alpha_i-\alpha_{0i}) - 2i\eta_i(\alpha_r-\alpha_{0r}) - \frac{1}{2}(\eta_r^2+\eta_i^2)}\, d\eta_r\, d\eta_i$$

$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty} e^{2i\eta_r(\alpha_i-\alpha_0) - \frac{1}{2}\eta_r^2}\, d\eta_r \int_{-\infty}^{\infty} e^{-2i\eta_i(\alpha_r-\alpha_{0r}) - \frac{1}{2}\eta_i^2}\, d\eta_i$$

Making use of the standard integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + iJx}\, dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{-J^2/2a}$$

we have

$$\int_{-\infty}^{\infty} e^{2i\eta_r(\alpha_i-\alpha_0) - \frac{1}{2}\eta_r^2}\, d\eta_r = \sqrt{2\pi}\, e^{-2(\alpha_i-\alpha_{0i})^2}$$

and

$$\int_{-\infty}^{\infty} e^{-2i\eta_i(\alpha_r-\alpha_{0r}) - \frac{1}{2}\eta_i^2}\, d\eta_i = \sqrt{2\pi}\, e^{-2(\alpha_r-\alpha_{0r})^2}$$

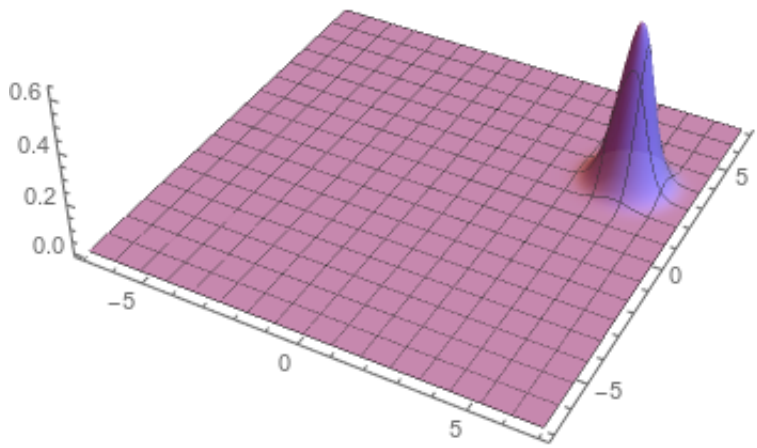
Finally, we can write the Wigner function for a Coherent state $|\alpha_0\rangle$ as

$$W(\alpha) = \frac{1}{\pi^2}\left[\sqrt{2\pi}\, e^{-2(\alpha_i-\alpha_{0i})^2}\sqrt{2\pi}\, e^{-2(\alpha_r-\alpha_{0r})^2}\right]$$

$$\therefore \boxed{W(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2}}$$

Having found this expression we now take a look at what such a Wigner function looks like with the example of $|5+3i\rangle$ which gives

$$W(\alpha) = \frac{2}{\pi} e^{-2|\alpha-(5+3i)|^2} = \frac{2}{\pi} e^{-2((\alpha_r-5)^2+(\alpha_i-3)^2)}$$



This is clearly a Gaussian function and a proper probability distribution, giving even more credence to the idea that coherent states are as classical as we can get in continuous variable quantum mechanics.

Wigner Function for an ensemble of two coherent states We have the state $\rho = \frac{1}{N}\left(|\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right)$. Let's normalise it, find its Wigner function and plot it for an example coherent state ensemble with $\alpha = 5+3i$. For normalisation, recall that density matrices have trace equal to 1 and observe the following properties for operators

$$\textbf{tr}(\mathbf{A}+\mathbf{B}) = \textbf{tr}(\mathbf{A}) + \textbf{tr}(\mathbf{B}) \qquad \textbf{tr}(c\mathbf{A}) = \textbf{tr}(\mathbf{A}) + c\textbf{tr}(\mathbf{A})$$

allowing one to write

$$\textbf{tr}(\rho) = 1$$
$$\textbf{tr}(\rho) = \frac{1}{N}\left(\textbf{tr}|\alpha_0\rangle\langle\alpha_0| + \textbf{tr}|-\alpha_0\rangle\langle-\alpha_0|\right) = 1$$

and since $|\pm\alpha_0\rangle$ is a pure state we have

$$(1+1) = N$$

and so

$$\rho = \frac{1}{2}\left(|\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right).$$

$$\chi_s(\eta) = \mathrm{tr}\left(\rho \hat{D}(\eta)\right) = \frac{1}{2}\int_{-\infty}^{\infty} \langle \alpha_0 | \left(|\alpha_0\rangle\langle\alpha_0| \hat{D}(\eta) + |-\alpha_0\rangle\langle -\alpha_0| \hat{D}(\eta)\right) |\alpha_0\rangle \, d^2\alpha_0$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \langle \alpha_0 | \hat{D}(\eta) | \alpha_0 \rangle + \langle \alpha_0 | -\alpha_0 \rangle\langle -\alpha_0 | \hat{D}(\eta) | \alpha_0 \rangle \, d^2\alpha_0$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \langle \alpha_0 | \hat{D}(\eta) | \alpha_0 \rangle + e^{-2|\alpha_0|^2} \langle -\alpha_0 | \hat{D}(\eta) | \alpha_0 \rangle \, d^2\alpha_0$$

Before considering the Wigner function as a whole, let's consider the form of the characteristic function so as to be able to fit it into the Wigner function in a way which is suitable. The Wigner function can be calculated now in the following way. So taking the example of $\rho = \frac{1}{2}\left(|5+3i\rangle\langle 5+3i| + |-5-3i\rangle\langle -5-3i|\right)$

$$W(\alpha) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{\eta^*\alpha - \alpha^*\eta}\left(\langle \alpha_0 | \hat{D}(\eta) | \alpha_0 \rangle + e^{-2|\alpha_0|^2} \langle -\alpha_0 | \hat{D}(\eta) | \alpha_0 \rangle\right) d^2\eta$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{\eta^*\alpha - \alpha^*\eta}\Big(\langle 0 | \hat{D}(-\alpha_0)\hat{D}(\eta)\hat{D}(\alpha_0) | 0 \rangle$$

$$+ e^{-2|\alpha_0|^2} \langle 0 | \hat{D}(\alpha_0)\hat{D}(\eta)\hat{D}(\alpha_0) | 0 \rangle\Big) d^2\eta$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{\eta^*\alpha - \alpha^*\eta}\left(e^{\frac{1}{2}(\alpha_0^*\eta - \eta^*\alpha_0)} e^{\frac{1}{2}((\alpha_0+\eta)^*(-\alpha_0) - (-\alpha_0)^*(\alpha_0+\eta))} e^{-\frac{|\eta|^2}{2}}\right.$$

$$\left. + e^{-2|\alpha_0|^2} e^{\frac{1}{2}(\alpha_0^*\eta - \eta^*\alpha_0)} e^{\frac{1}{2}((\alpha_0+\eta)^*\alpha_0 - \alpha_0^*(\alpha_0+\eta))} e^{-\frac{1}{2}|\eta+2\alpha_0|^2}\right) d^2\eta$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{\eta^*\alpha - \alpha^*\eta}\left(e^{\alpha_0^*\eta - \eta^*\alpha_0 - \frac{|\eta|^2}{2}} + e^{-2|\alpha_0|^2} e^{-\frac{1}{2}|\eta+2\alpha_0|^2}\right) d^2\eta$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{\eta^*(\alpha-\alpha_0) - \eta(\alpha-\alpha_0)^* - \frac{|\eta|^2}{2}} + e^{\eta^*(\alpha+\alpha_0) - \eta(\alpha-\alpha_0)^* - 4|\alpha_0|^2 - \frac{|\eta|^2}{2}} \, d^2\eta$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{(\eta_r - i\eta_i)((\alpha_r-\alpha_{0r}) + i(\alpha_i - \alpha_{0i})) - (\eta_r + i\eta_i)((\alpha_r - \alpha_{0r}) - i(\alpha_i - \alpha_{0i})) - \frac{(\eta_r^2+\eta_i^2)}{2}}$$

$$+ e^{(\eta_r - i\eta_i)((\alpha_r+\alpha_{0r}) + i(\alpha_i+\alpha_{0i})) - (\eta_r + i\eta_i)((\alpha_r - \alpha_{0r}) - i(\alpha_i - \alpha_{0i})) - 4(\alpha_{0r}^2+\alpha_{0i}^2) - \frac{(\eta_r^2+\eta_i^2)}{2}} \, d^2\eta$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} e^{2i\eta_r(\alpha_i - \alpha_{0i}) - 2i\eta_i(\alpha_r - \alpha_{0r}) - \frac{(\eta_r^2+\eta_i^2)}{2}} + e^{2\eta_r(\alpha_{0r}+i\alpha_{0i}) + 2\eta_i(\alpha_{0i} - i\alpha_{0r}) - 4(\alpha_{0r}^2+\alpha_{0i}^2) - \frac{(\eta_r^2+\eta_i^2)}{2}} \, d^2\eta$$

Making use of the standard integrals

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + iJx} \, dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{-J^2/2a}$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + Jx} \, dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a}$$

we have

$$= \frac{1}{2\pi^2}\Big[\left(\sqrt{2\pi}\, e^{-2(\alpha_i - \alpha_{0i})^2}\right)\left(\sqrt{2\pi}\, e^{-2(\alpha_r - \alpha_{0r})^2}\right)$$

$$+ \left(\sqrt{2\pi}\, e^{-2(\alpha_r^2 - 2\alpha_{0r}\alpha_r + \alpha_{0r}^2)}\right)\left(\sqrt{2\pi}\, e^{-2(\alpha_i^2 + 2\alpha_{0i}\alpha_i + \alpha_{0i}^2)}\right)\Big]$$

$$= \frac{1}{2\pi^2}\left[2\pi e^{-2|\alpha-\alpha_0|^2} + 2\pi e^{-2(\alpha_r^2+\alpha_{0r}^2+\alpha_i^2+\alpha_{0i}^2)}\right]$$

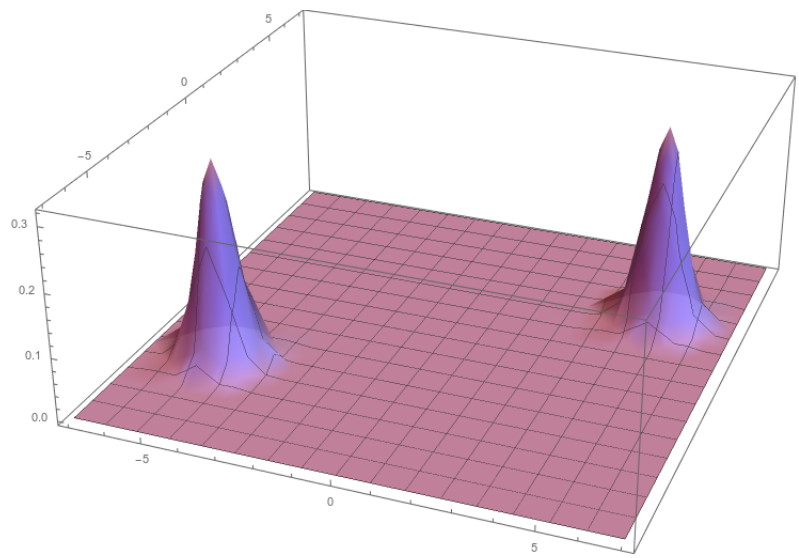
$$= \frac{1}{\pi}\left(e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2}\right)$$

$$\boxed{\therefore\ W_\rho(\alpha) = \frac{1}{\pi}\left(e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2}\right)}$$

we have

$$W(\alpha) = \frac{1}{\pi}\left(e^{-2\left((\alpha_r-5)^2+(\alpha_i-3)^2\right)} + e^{-2\left((\alpha_r+5)^2+(\alpha_i+3)^2\right)}\right)$$

giving the following plot.



The Wigner Function for a Cat State A cat state is defined as

$$|\text{🐱}\rangle = \frac{1}{N}\left(|\alpha_0\rangle + e^{i\theta}\,|-\alpha\rangle\right) \qquad\qquad \langle\text{🐶}| = \frac{1}{N}\left(\langle\alpha_0| + e^{-i\theta}\,\langle-\alpha|\right)$$

where its bra $\langle$🐶$|$ is the dog linear functional.[12] These states are referred to as *cat* states because they are a superposition of *classical* light with opposite phases which is reminiscent of the Schrödinger's famous *Gedanken* experiment involving a cat in a superposition of between being dead and alive.

So as to normalise this state consider

$$\begin{aligned}
\langle\text{🐶}|\text{🐱}\rangle &= 1 \\
\implies \frac{1}{N^2}\left(\langle\alpha_0|\alpha_0\rangle + e^{i\theta}\,\langle\alpha_0| - \alpha_0\rangle + e^{-i\theta}\,\langle-\alpha_0|\alpha_0\rangle + \langle-\alpha_0|\alpha_0\rangle\right) &= 1 \\
2 + e^{i\theta}\,\langle 0|\,\hat{D}(-\alpha_0)\hat{D}(-\alpha_0)\,|0\rangle + e^{i\theta}\,\langle 0|\,\hat{D}(\alpha_0)\hat{D}(\alpha_0)\,|0\rangle &= N^2 \\
2 + e^{i\theta}e^{\frac{1}{2}\left(\alpha_0^*\alpha_0 - \alpha_0^*\alpha_0 - |2\alpha_0|^2\right)} + e^{-i\theta}e^{\frac{1}{2}\left(\alpha_0^*\alpha_0 - \alpha_0^*\alpha_0 - |2\alpha_0|^2\right)} &= N^2 \\
2 + e^{i\theta}e^{-2|\alpha_0|^2} + e^{-i\theta}e^{-2|\alpha_0|^2} &= N^2 \\
2\left(1 + e^{-2|\alpha_0|^2}\frac{\left(e^{i\theta} + e^{-i\theta}\right)}{2}\right) &= N^2 \\
\therefore\ \sqrt{2\left(1 + e^{-2|\alpha_0|^2}\cos\theta\right)} &= N.
\end{aligned}$$

Before one starts calculating the Wigner function let's take a look at the characteristic function to see how

[12] Of course, the hermitian adjoint of a cat is a dog.

to include it in the calculation. The density matrix for a cat state can be written as

$$\rho_{|\text{😸}\rangle} = \frac{1}{2\left(1+e^{-2|\alpha_0|^2}\cos\theta\right)}\left(|\alpha_0\rangle\langle\alpha_0| + e^{i\theta}|-\alpha_0\rangle\langle\alpha_0| + e^{-i\theta}|\alpha_0\rangle\langle-\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right)$$

Calculating the Wigner function we proceed as follows

$$\rho_{|\alpha\rangle} = \frac{1}{2(1+e^{-2|\alpha_0|^2}\cos\theta)}\left[|\alpha_0\rangle\langle\alpha_0| + e^{i\theta}|-\alpha\rangle\langle\alpha_0| + e^{-i\theta}|\alpha_0\rangle\langle-\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right]$$

$$\chi_s(\eta) = tr\left(\rho\hat{D}(\eta)\right) = \frac{1}{N^2}\int_{-\infty}^{\infty}\langle\alpha_0|\alpha\rangle\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{i\theta}\langle\alpha_0|-\alpha\rangle\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-i\theta}\langle\alpha_0|\alpha\rangle\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + \langle\alpha_0|-\alpha\rangle\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\, d^2\alpha$$

$$= \frac{1}{N^2}\int_{-\infty}^{\infty}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{i\theta}e^{-\frac{|\alpha_0|^2}{2}}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-i\theta}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-|\alpha_0|^2/2}\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\, d^2\alpha$$

$$= \frac{1}{N^2}\int_{-\infty}^{\infty}\left(1+e^{i\theta-\frac{|\alpha_0|^2}{2}}\right)\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + \left(e^{-i\theta}+e^{-|\alpha_0|^2/2}\right)\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\, d^2\alpha$$

$$\Rightarrow W(\alpha) = \frac{1}{N^2\pi^2}\int_{-\infty}^{\infty}e^{\eta^*\alpha-\alpha^*\eta}\left[\left(1+e^{i\theta-\frac{|\alpha_0|^2}{2}}\right)\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + \left(e^{-i\theta}+e^{-|\alpha_0|^2/2}\right)\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\right]d^2\eta$$

$$= \frac{1}{N^2\pi^2}\int_{-\infty}^{\infty}\boxed{e^{\eta^*\alpha-\alpha^*\eta}\left[\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-|\alpha_0|^2/2}\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\right]} + e^{\eta^*\alpha+\alpha^*\eta}\left[e^{i\theta-\frac{|\alpha_0|^2}{2}}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-i\theta}\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\right]d^2\eta$$

But this is just the integral from the Wigner function in part a) for $\rho = \frac{1}{2}\left[|\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right]$ so we may write

$$\frac{2}{\pi N}\left(e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2}\right) + \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*\alpha+\alpha^*\eta}\left(e^{i\theta-\frac{|\alpha_0|^2}{2}}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-i\theta}\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\right)d^2\eta$$

Focusing on the unevaluated integral

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*\alpha-\alpha^*\eta}\left(e^{i\theta-\frac{|\alpha_0|^2}{2}}\langle\alpha_0|\hat{D}(\eta)|\alpha_0\rangle + e^{-i\theta}\langle-\alpha_0|\hat{D}(\eta)|\alpha_0\rangle\right)d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*\alpha-\alpha^*\eta}\left(e^{i\theta-\frac{|\alpha_0|^2}{2}}e^{\frac{1}{2}\left[(\eta^*-\eta^*)(\alpha_0)-(-\alpha_0)^*(\alpha_0-\eta)\right]}e^{-\frac{|\eta|^2}{2}} + e^{-i\theta}e^{\frac{1}{2}\left[(\alpha_0^*-\eta^*)(-\alpha_0)-(-\alpha_0)^*(\alpha_0-\eta)\right]}\right)d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*\alpha-\alpha^*\eta}\left(e^{i\theta-\frac{|\alpha_0|^2}{2}}e^{\alpha_0^*\eta-\eta^*\alpha_0-\frac{|\eta|^2}{2}} + e^{-i\theta-\frac{1}{2}|\eta+2\alpha_0|^2}\right)d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*(\alpha-\alpha_0)-\eta(\alpha-\alpha_0)^*-\frac{|\alpha_0|^2}{2}-\frac{|\eta|^2}{2}+i\theta} + e^{\eta^*\alpha-\alpha^*\eta-i\theta-\frac{1}{2}|\eta+2\alpha_0|^2}d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*(\alpha-\alpha_0)-\eta(\alpha-\alpha_0)^*-\frac{|\eta|^2}{2}-\frac{|\alpha_0|^2}{2}+i\theta} + e^{\eta^*\alpha-\alpha^*\eta-i\theta-\frac{1}{2}(\eta+2\alpha_0)^*(\eta+2\alpha_0)}d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*(\alpha-\alpha_0)-\eta(\alpha-\alpha_0)^*-\frac{|\eta|^2}{2}-\frac{|\alpha_0|^2}{2}+i\theta} + e^{\eta^*\alpha-\alpha^*\eta-i\theta-\frac{1}{2}\left(|\eta|^2+4|\alpha_0|^2+2\alpha_0\eta^*+2\alpha_0^*\eta\right)}d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{\eta^*(\alpha-\alpha_0)-\eta(\alpha+\alpha_0)^*-i\theta-\frac{|\eta|^2}{2}-2|\alpha_0|^2} + e^{\eta^*(\alpha-\alpha_0)-\eta(\alpha+\alpha_0)-2\eta(\alpha_0-i\alpha_0)-2(\alpha_0^2+\alpha_0^2)-(\eta^2+\eta^2)-i\theta_r+\theta_i}d^2\eta$$

$$= \frac{1}{\pi^2N^2}\int_{-\infty}^{\infty}e^{2\eta_r(\alpha_r-\alpha_{0r})-2i\eta_i(\alpha_r-\alpha_{0r})-\frac{(\eta_r^2+\eta_i^2)}{2}-\frac{(\alpha_{0r}^2+\alpha_{0i}^2)}{2}+i\theta_r-\theta_i} + e^{2\eta_r(\alpha_{0r}+i\alpha_{0i})-2i\eta_i(\alpha_{0r}-i\alpha_{0i})-2(\alpha_{0r}^2+\alpha_{0i}^2)-(\eta_r^2+\eta_i^2)-i\theta_r+\theta_i}d^2\eta$$

$$= \frac{1}{\pi^2N^2}\left(\int_{-\infty}^{\infty}e^{2i\eta_r(\alpha_i-\alpha_{0i})-\frac{\eta_r^2}{2}-\frac{\alpha_{0r}^2}{2}+i\theta_r}d\eta_r\int_{-\infty}^{\infty}e^{2i\eta_i(\alpha_r-\alpha_{0r})-\frac{\eta_i^2}{2}-\frac{\alpha_{0i}^2}{2}-\theta_i}d\eta_i\right.$$

$$\left.+\int_{-\infty}^{\infty}e^{2\eta_r(\alpha_{0r}+i\alpha_i)-2\alpha_{0r}^2-\frac{\eta_r^2}{2}-i\theta_r}d\eta_r\int_{-\infty}^{\infty}e^{2\eta_i(\alpha_i-i\alpha_r)-2\alpha_{0i}^2-\frac{\eta_i^2}{2}+\theta_i}d\eta_i\right)$$

$$= \frac{1}{\pi^2N^2}\left[\sqrt{2\pi}\,e^{-2(\alpha_i-\alpha_{0i})^2-2\alpha_{0r}^2+i\theta_r}\cdot\sqrt{2\pi}\,e^{-2(\alpha_r-i\alpha_r)^2-2\alpha_{0i}^2+\theta_i}\right.$$

$$\left.+\sqrt{2\pi}\,e^{2(\alpha_{0r}+i\alpha_i)^2-2\alpha_{0r}^2-i\theta_r}\cdot\sqrt{2\pi}\,e^{2(\alpha_{0i}-i\alpha_r)^2-2\alpha_{0i}^2+\theta_i}\right]$$

$$= \frac{1}{\pi^2N^2}\left[2\pi e^{-2|\alpha-\alpha_0|^2-2|\alpha_0|^2}e^{i\theta} + 2\pi e^{|\alpha+\alpha_0|^2-2|\alpha_0|^2}e^{-i\theta}\right]$$

So we have

$$W(\alpha) = \frac{2}{\pi(1+e^{-2|\alpha_0|^2}\cos\theta)}\left[e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha-\alpha_0|^2-2|\alpha_0|^2}e^{i\theta} + e^{-2|\alpha+\alpha_0|^2-2|\alpha+\alpha_0|^2-2|\alpha_0|^2}e^{-i\theta}\right]$$

$$= \frac{2}{\pi(1+e^{-2|\alpha_0|^2}\cos\theta)}\left[e^{-2|\alpha-\alpha_0|^2}\left(1+e^{-2|\alpha_0|^2}e^{i\theta}\right) + e^{-2|\alpha+\alpha_0|^2}\left(1+e^{-2|\alpha_0|^2}e^{-i\theta}\right)\right]$$

$$\boxed{W(\alpha) = \frac{2}{\pi(1+e^{-2|\alpha_0|^2}\cos\theta)}\left[e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} + 2e^{-2|\alpha|^2}\cos\left(2(\alpha_r\alpha_{0r}+\alpha_i\alpha_{0i})+\theta\right)\right]}$$

Examining the resulting plot we see that this distribution has a negative values and clearly has non-Gaussian portions. The two Gaussians, represent the probability of being in the coherent states where as the non-Gaussian portion refers to the interference in the superposition of these two coherent states.

2.4 The Covariance Matrix Representation

Having worked with Characteristic functions and probability densities, it's now time to work with another formulation of probability distributions for distributions that require only their first and second moments to be described, the covariance matrix. Recalling our vector of operators which describe states in our Hilbert Space, $\hat{R} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_n, \hat{p}_n)$, we observe that the first and second moments of a state can be

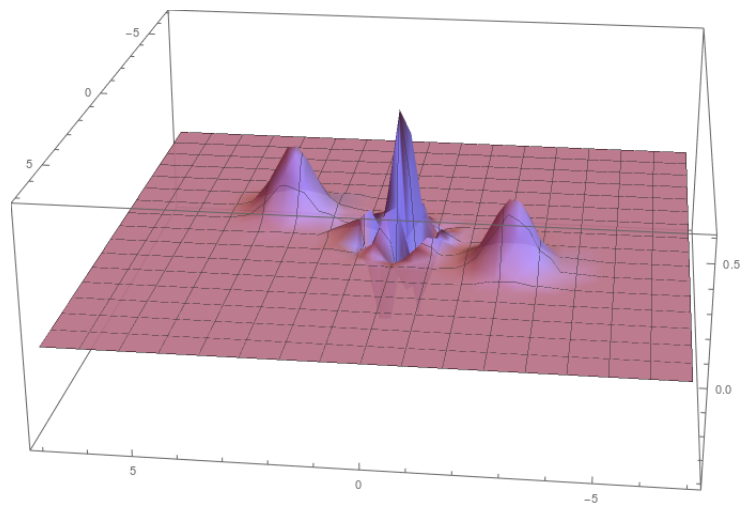


Figure 2: The Wigner Function for a Cat State

given as a function of the expectation values of the vector of quadrature operators as

$$d_j = \langle \hat{R}_j \rangle_\rho \qquad\qquad \sigma_{ij} = \frac{1}{2}\langle \hat{R}_i\hat{R}_j - \hat{R}_j\hat{R}_j \rangle_\rho + \langle \hat{R}_i \rangle_\rho \langle \hat{R}_j \rangle_\rho$$

respectively. As such for $j = 1$ we may write the covariance matrix as

$$\hat{\sigma}(\hat{x}, \hat{p}) = \begin{pmatrix} \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 & \frac{1}{2}\langle \{\hat{x}, \hat{p}\} \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle \\ \frac{1}{2}\langle \{\hat{p}, \hat{x}\} \rangle - \langle \hat{p} \rangle \langle \hat{x} \rangle & \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \end{pmatrix}$$

which is real, symmetric and positive definite.

Now recalling what we learned in the first section, covariance matrices can be used to formulate multivariable probability densities and as such, we can write Gaussian Wigner Functions in terms of a covariance matrix in the following way

$$W(\hat{R}) = \frac{1}{\pi\sqrt{|\hat{\sigma}|}} e^{-(\hat{R}-\hat{d})^T \sigma^{-1} (\hat{R}-\hat{d})}$$

where $\hat{d}$ is the vector of first moments.

For a coherent state we have $\hat{d} = \sqrt{2}\begin{bmatrix} \Re(\alpha_0) \\ \Im(\alpha_0) \end{bmatrix}$ and $\hat{\sigma} = \hat{I}$ clearly allowing one to recover

$$W(\alpha) = \frac{2}{\pi} e^{-2|\alpha - \alpha_0|^2}.$$

Robertson-Schrödinger Uncertainty Relation

Entanglement with Gaussian States

The Lyapunov Equation