Work on PHY1145 - Assignment 3 - Electrostatics 2

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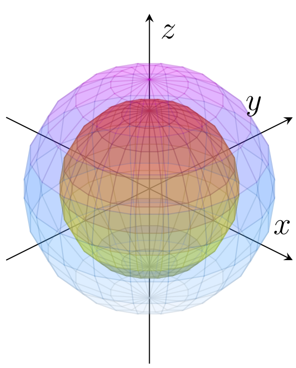
April 27, 2018

Problem 1

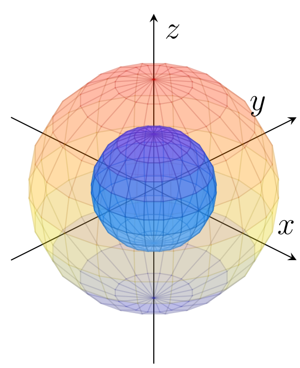
An insulated solid sphere of radius a has a uniform charge density ρ. Compute the electric potential everywhere (inside and outside the sphere).

Solution 1

Firstly the problem must be contextualised. Some charge is distributed on an insulated solid sphere of radius a resulting in a uniform charge density ρ throughout the sphere. To calculate the electric potential what first must be calculated is the electric field generated by the charge distribution.
Given that it is a symmetric charge density and that the electric field produced will be radial then it will be a good idea to make use of Gauss' Law.
We are tasked to find the potential at two different positions, inside and outside the sphere and thus two Gaussian Surface configurations will be constructed referred two as below (a) and (b) where the Gaussian Surface is represented using a cool colormap and the insulated sphere charge distributed on it is represented using a hot colormap.



(a) Gaussian Surface outside with charged sphere inside



(b) Gaussian Surface inside with charged sphere outside

In case (a) since we want to find the charge outside the sphere, a Gaussian surface of radius $r \geq a$ is constructed such that by Gauss' Law the field at the point of the Gaussian surface ie: outside the sphere will be given by the charge inside this closed surface, the charged insulated sphere.
In case (b) since we want to find the charge inside the sphere, a Gaussian surface of radius $r \leq a$ is constructed such that by Gauss' Law the electric field inside the sphere will be captured by the charge found within this Gaussian surface.

The electric field for each case is found as follows.
By Gauss' Law the electric flux at the point of the the Gaussian surface is equal to the charge inside the closed surface divided by the permittivity

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0} \tag{1}$$

By the fact that the field is radial and symmetry

$$\Phi = E \int dA = \frac{Q}{\epsilon_0} \tag{2}$$

$$\Phi = EA = \frac{Q}{\epsilon_0} \tag{3}$$

Now this we can work with. Everything is known except for Q in this preceeding statement but this can easily by overcome when remembering that the charge density ρ is known.
Up until this point the working is identical for the two cases but now Q inside will be different and how much of ρ will be "caught" by the Gaussian Surface will differ.

$$Q = \int_0^a dq \text{ but } dq = \rho dV \qquad Q = \int_0^r dq \text{ but } dq = \rho dV \tag{4}$$

$$Q = \int_0^a \rho dV \qquad Q = \int_0^r \rho dV \tag{5}$$

$$Q = \rho \int_0^a dV \qquad Q = \rho \int_0^r dV \tag{6}$$

$$Q = \rho V_a \qquad Q = \rho V_r \tag{7}$$

$$Q = \rho \frac{4\pi a^3}{3} \qquad Q = \rho \frac{4\pi r^3}{3} \tag{8}$$

Substituting this back into (3) E can now be found.

$$E 4\pi r^2 = \frac{Q}{\epsilon_0} = \rho \frac{4\pi a^3}{3\epsilon_0} \qquad E 4\pi r^2 = \frac{Q}{\epsilon_0} = \rho \frac{4\pi r^3}{3\epsilon_0} \tag{9}$$

$$E_a = \frac{\rho a^3}{3\epsilon_0 r^2} \qquad E_b = \frac{\rho r}{3\epsilon_0} \tag{10}$$

Now that the Electric field at these points has been found we may calculate the electric potential at these points using these values.
Electric Potential can be thought of as the work required to bring a point charge to given location from infinity. Mathematically we illustrate this as follows

$$V(r) = -\int_\infty^r \vec{E} dr'$$

So restating the electric field magnitudes in their vectorial forms in terms of r by using the relationship between ρ and Q as seen in (8) all that need be done is integration.

$$Q = \rho V$$

Where in this case we are defining ρ for the whole sphere and not considering a section of it as was done in case b

$$\rho = \frac{3Q}{4\pi a^3}$$

$$\begin{cases} E_a = \dfrac{\rho a^3}{3\epsilon_0 r^2} \\ E_b = \dfrac{\rho r}{3\epsilon_0} \end{cases} \implies \begin{cases} E_a = \dfrac{Q}{4\pi\epsilon_0 r^2} \\ E_b = \dfrac{Qr}{4\pi\epsilon_0 a^3} \end{cases} \implies \begin{cases} \vec{E}_a = \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} \\ \vec{E}_b = \dfrac{Qr}{4\pi\epsilon_0 a^3}\hat{r} \end{cases}$$

Now all that is left is to integrate.
Starting with V_{out}, the limits will be infinity and r as we wish to place the test charge at the point when the Gaussian Surface has a radius r larger than a thus bringing a test charge from infinity to the equipotential at radius r.

$$V_{\text{out}}(r) = -\int_{\infty}^{r} \vec{E}_a dr$$

$$V_{\text{out}}(r) = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$\underline{\underline{V_{\text{out}}(r) = \frac{Q}{4\pi\epsilon_0 r}}}$$

Now for the case of V_{in} it gets perhaps slightly more complicated as the test charge must first be taken from infinity to the surface of the sphere, involving the electric field outside the sphere. After this the test charge must the be taken inside the sphere from its surface involving the electric field inside the sphere. So first we will have an integral with limits from infinity to a and then an integral with limits from a to r. The sum of these two integrals will be the desired answer.
It is good to note that the integral may not be joined as the function of the electric field changes.

$$V_{\text{in}}(r) = V_{\infty\to a} + V_{a\to r} = -\int_{\infty}^{a} \vec{E}_a dr - \int_{a}^{r} \vec{E}_b dr$$

$$V_{\text{in}}(r) = -\int_{\infty}^{a} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{a}^{r} \frac{Qr}{4\pi\epsilon_0 a^3} dr$$

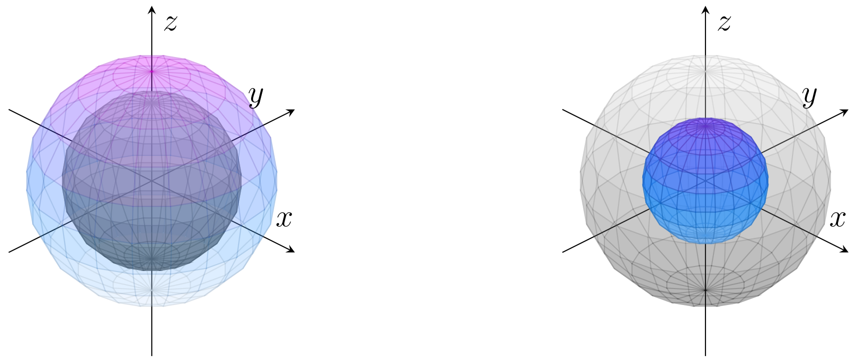
$$V_{\text{in}}(r) = \frac{Q}{4\pi\epsilon_0 a} - \Big|_a^r \frac{Qr^2}{8\pi\epsilon_0 a^3}$$

$$\underline{\underline{V_{\text{in}}(r) = \frac{3Q}{8\pi\epsilon_0 a} - \frac{Qr^2}{8\pi\epsilon_0 a^3}}}$$

Problem 2

If the electric field in some region of space is zero, does it imply that there is no electric charge in that region?

Solution 2



(a) Gaussian Surface outside with charged **hollow** sphere inside

(b) Gaussian Surface inside with charged **hollow** sphere outside

No. Consider the situation above where a hollow sphere possesses a charged surface. To find the electric potential at this point we consider Gauss' Law and think about whether there are any charges within this closed surface either by placing a Gaussian surface within it as in (b) or by considering in a straight forward manner that the sphere is hollow meaning that it has no charges inside it. From this we see that whilst the surface is charged it still has an electric field which is 0.
The converse of the would be a non-charged region possessing an electric field value which is the case for the region outside of the sphere found by the approach depicted in diagram (a) and supported once more by Gauss' Law where the Gaussian surface itself has no charge but contains a charge and as a result posses an Electric field value.

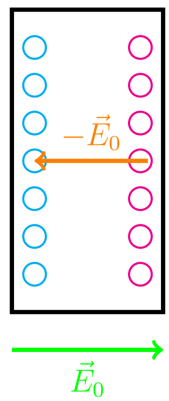
Problem 3

Why should electrostatic field be zero inside a conductor?

Solution 3

The structure of a conductor is defined by the presence of free electrons which typically form within the process of metallic bonding such that the formation of ions requires that the external valence electrons which are preventing an octet valence shell be liberated from the valence shell of the metal atoms forming metal ions and a surrounding sea of free electrons which roam what is known as the conduction band.
Now the presence of a field would result in the free electrons acquiring some drift velocity and so we exit the realm of electrostatics but this is a cop out answer.
Say we place a conductor in some field $\vec{E}_0$, and as described above it is understood that the only motive charged particles within an atomic structure are the free electrons, as the name implies, and given that the electrostatic field has similar charge to that of the electrons then the electrons in the structure will be repelled to one side of the conductor

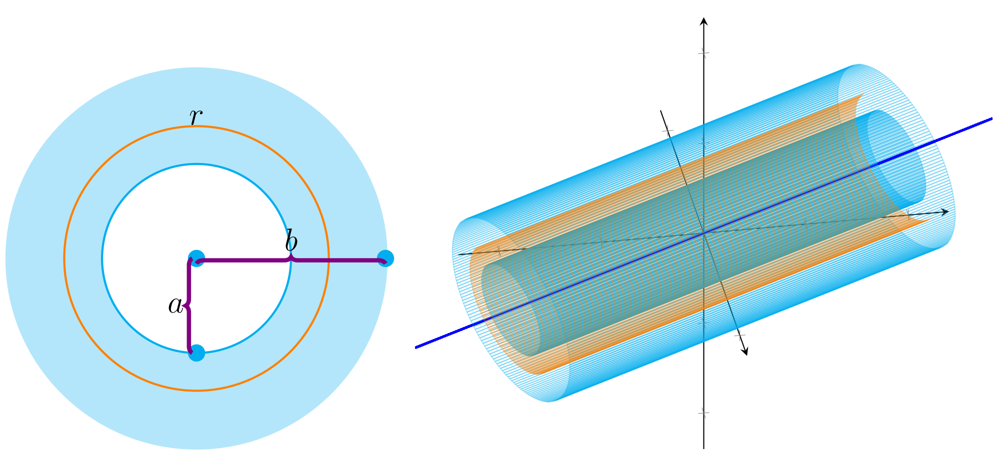
creating a partially positively charged area by induction (this would be the side of the holes within electron-hole theory).
Now, given that the electrons move in a direction opposing the field by repulsion, as a result an electric field equal to $\vec{E}_0$ but opposite in direction is created which **cancels out the effective field**. Meaning that no net field can be formed inside a conductor.



Problem 4

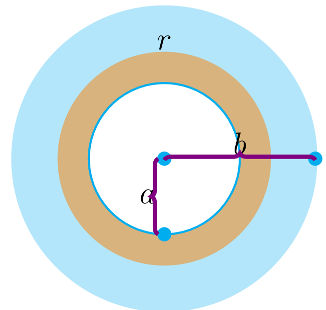
Consider a **non-conducting** cylinder of infinite length with a hollow core. The inner radius is a, the outer radius is b, and the solid region in between carries a uniformly-distributed volume charge density ρ.

Using Gauss Law, calculate the electric field at a radius of r from the axis of the cylinder, where $a < r < b$.



(a) Cross - Sectional View (b) 3-D View

Solution 4



Right, so cylindrical charge distributions are closed surfaces and as such the case will be attacked through the strategy of Gauss' Law and Gaussian surfaces.
What we care about is the hollow tube in orange depicted on the right and as Gaussian surface we can consider a tube exactly fitted on to the tube at r say of radius $r + \epsilon$.

Now to start getting into the thick of it we recall that flux is given by the following relation which used in conjunction to Gauss' Law will be used for the determination of $\vec{E}_r$.

$$\Phi = \oint \vec{E} \cdot dA = \frac{Q}{\epsilon_0}$$

By the fact that the field is radial and symmetry

$$\Phi = E \int dA = \frac{Q}{\epsilon_0}$$
$$\Phi = EA = \frac{Q}{\epsilon_0}$$

To move on Q must be found by extracting it from ρ

$$Q = \int_a^r dq \text{ but } dq = \rho dV$$
$$Q = \rho \int_a^r dV$$
$$Q = \rho V \Big|_a^r = \rho \pi r'^2 l \Big|_a^r$$
$$Q = \rho\pi(r^2 - a^2)l$$

Making use of the surface area of a hollow cylinder and substituting the achieved Q into Φ

$$\implies \Phi = E 2\pi r l = \frac{\rho\pi(r^2 - a^2)l}{\epsilon_0}$$
$$\therefore E = \frac{\rho(r^2 - a^2)}{2r\epsilon_0}$$

Problem 5

Suppose the electric potential due to a certain charge distribution can be written in Cartesian coordinates as

$$V(x, y, z) = Ax^2y^2 + Bx$$

where A, B and C are constants. What is the associated electric field?

Solution 5

It is understood that the electric potential may be defined mathematically as follows

$$V(r) = -\int_{\infty}^{r} \vec{E} dr'$$

with some basic knowledge of vector calculus we can reverse this argument to achieve the electric field as the gradient of the potential scalar field.

$$\implies \vec{E} = -\vec{\nabla} V(r)$$

Applying this relation the case at hand

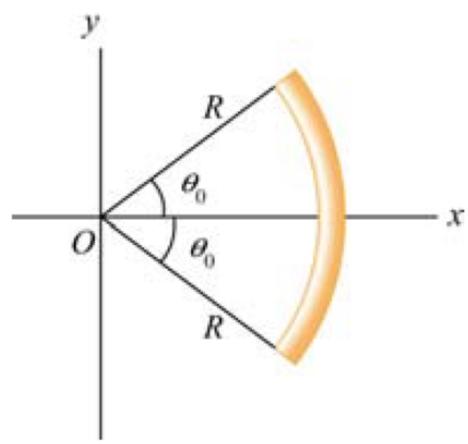
$$\vec{E} = -\vec{\nabla} V(x, y, z) = -\left(\frac{\partial(Ax^2y^2 + Bx)}{\partial x}\hat{i} + \frac{\partial(Ax^2y^2 + Bx)}{\partial y}\hat{j} + \frac{\partial(Ax^2y^2 + Bx)}{\partial z}\hat{k}\right)$$

$$\vec{E} = -(2Ay^2x + B)\hat{i} - (2Ax^2y)\hat{j}$$

Problem 6

A thin rod with a uniform charge per unit length λ is bent into the shape of an arc of a circle of radius R. The arc subtends a total angle 2θ, symmetric about the x-axis, as shown in Figure.
What is the electric field $\vec{E}$ at the origin O?

Figure 1: *This figure is not work of the author*



Solution 6

This is not a closed charge distribution meaning that Gauss' Law cannot be used but none the less it is continuous meaning that by breaking the charge down into some elementary charge dq which would contribute some $d\vec{E}$ which when integrated will give $\vec{E}$

$$\vec{E} = \frac{\vec{F}}{Q}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{P}} \frac{1}{R^2}\hat{r}\, dq$$

First thing's first let's sort out dq. Now, this nothing more than a bent line charge of charge density λ and length of arc $R\theta$, so for an infinitessimal piece

$$\implies dq = \lambda R d\theta$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{P}} \frac{\lambda}{R}\hat{r}\, d\theta$$

The integral is now evidently in terms of an angle and rightly so given that we have a circular charge density but as a result $\hat{r}$ must be decomposed to its polar form so that it can be used.

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

If $\hat{r}$ is left in this typical statement then it will point from the origin to the infinitessimal charge in the charge density and it is obvious that the electric field we desire will point in the opposite direction, outwards from the charge density.

$$\implies \hat{r} = -\cos\theta\hat{i} - \sin\theta\hat{j}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{P}} \frac{\lambda}{R}(-\cos\theta\hat{i} - \sin\theta\hat{j})\, d\theta$$

Now the limits of the integral must be chosen and given that we wish to span the arc, $-\theta$ and θ are chosen

$$\vec{E} = \frac{\lambda}{4\pi R\epsilon_0}\int_{-\theta}^{\theta} (-\cos\theta\hat{i} - \sin\theta\hat{j})\, d\theta$$

$$\vec{E} = \frac{\lambda}{4\pi R\epsilon_0}(-\sin\theta\hat{i} + \cos\theta\hat{j})\Big|_{-\theta}^{\theta}$$

$$\vec{E} = \frac{\lambda}{4\pi R\epsilon_0}\left[(-\sin\theta\hat{i} + \cos\theta\hat{j}) - (-\sin(-\theta)\hat{i} + \cos(-\theta)\hat{j})\right]$$

$$\because \sin(-x) = -\sin x \text{ and } \cos(-x) = \cos(x)$$

$$\therefore \underline{\underline{\vec{E} = \frac{\lambda}{4\pi R\epsilon_0}(-2\sin\theta\hat{i})}}$$