Notes on MAT1211 - Analysis 1

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Abstract

These notes will be based off of the lectures from MAT1211 given by Dr.Beatriz Zamora at the University of Malta within the Winter and Spring of 2018. These notes will be heavily involved with the material found in the book by Spivak, Calculus and even more so Abbott, Understanding Analysis.

The **motivation** behind these notes is to recapitualise in a more formal manner what is discussed in the lectures and to compile information on the topics from multiple sources making for a more complete reference. These notes are an exercise of my personal thoughts and should in no way be considered official or affiliated with the University other than the fact that I attended these lectures. *Any comments or error-pointing related to these notes are to be directed at* **jqed.xuereb@gmail.com**

Contents

Chapter 1

Preliminaries and Introductory Considerations

1.1 The Real Ordered Field

1.1.1 What does it mean to be a "Real Ordered Field"

Field a set F on which two **binary operations** say $\{\cdot, +\}$ are defined satisfying the following properties $\forall a, b, c \in F$

1. $a + b \in F$ - Additive Closure
2. $a \cdot b \in F$ - Multiplicative Closure
3. $(a + b) + c = a + (b + c)$ - Associativity of Addition
4. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ - Associativity of Multiplication
5. $\exists 0 \in F : i)a + 0 = a$ - Null element
6. $\exists 0 \in F : ii)\exists b : a + b = 0$ - Additive Inverse element
7. $\exists 1 \in F : ii)a \cdot 1 = 1$ - Multiplicative Identity element
8. $\exists 1 \in F : ii)\exists b : a \cdot b = 1$ - Multiplicative Inverse element
9. $a + b = b + a$ - Commutativity of Addition
10. $a \cdot b = b \cdot a$ - Commutativity of Multiplication
11. $a \cdot (b + c) = a \cdot b + a \cdot c$ - Distributivity of Multiplication over Addition

The binary operations $\{\cdot, +\}$ where chosen for familiarity but a field may be constituted by any other operations satisfying the conditions specified above and for this reason the inverses may be written out using the notation $-a$ and a^{-1} respectively.

The Real Numbers are a **field** but also possess the quality of **Order** meaning that they are known as an **Ordered Field** more than that a complete ordered field but more on this below. **Order** is a quality possessed by fields which satisfy the following axioms,

1. $\forall a \in F,\ a = 0 \sqcup a \in P \sqcup -a \in P$ - Trichotomy
2. $a, b \in P \implies a + b \in P$ - Closure under addition
3. $a, b \in P \implies a \cdot b \in P$ - Closure under multiplication

Where P is the subset of positive elements of F. Note that the notation $\sqcup$ is the disjunctor and infers and mutually exclusive OR where as $\vee$ is the mutually inclusive OR.

Within the context of $\mathbb{R}$ the positive subset, P, is $\mathbb{R}^+$. With this notion at play, **inequalities** may be defined in $\mathbb{R}^+$

$$x < y \iff \exists k \in \mathbb{R}^+ : x + k = y \qquad\qquad x \geq y \iff x > y \vee x = y$$

The less than inequality applies the same definitions in reversal. With this definition the following relations follow by applying the definition with the properties that define **Order**.

1. $a = b \sqcup a < b \sqcup a > b$ - Trichotomy
2. $a \geq b \wedge b \geq a \implies a$ - Antisymmetry
3. $a > b \wedge b > c \implies a > c$ - Transitivity
4. $a > b \implies a + c > b + c$
5. $a > b \wedge c > 0 \implies a \cdot c > b \cdot c$

1.1.2 Inequalities of relevance

After now establishing order and having defined $>$ and $\geq$ it is now a good point to introduce inequalities which will prove to be most important for the purpose of real and complex analysis given that such matters deal with limits and the definition of a limit, as shall be seen later on, makes use of inequalities. For this reason many proofs in analysis make use

Absolute Value

$$|a| = \begin{cases} a & a \geq 0 \\ -a & a \leq 0 \end{cases}$$

It should be noted that the fact that the absolute value within its nature involves cases that proofs related to it are to be proved in a proofs by cases manner.
Proposition. $|a \cdot b| = |a| \cdot |b|$

Proof. Given that the absolute value involves cases then the proof to shall follow by cases and thus involves four cases where $a, b \neq 0$ as this would give the trivial result.

1. $a > 0, b > 0$
2. $a < 0, b > 0$
3. $a > 0, b < 0$
4. $a < 0, b < 0$

For 1.

$$\begin{aligned} a > 0 \implies |a| = a \wedge b > 0 \implies |b| = b & \quad \text{by definition of absolute value} \qquad (1.1) \\ a \cdot b > 0 & \quad \text{since } a \text{ and } b \text{ are positive} \qquad (1.2) \\ \implies |a \cdot b| = a \cdot b & \quad \text{by definition of absolute value} \qquad (1.3) \\ \implies |a \cdot b| = |a||b| & \quad \text{by (1)} \qquad (1.4) \end{aligned}$$

For 2. and similarly 3.

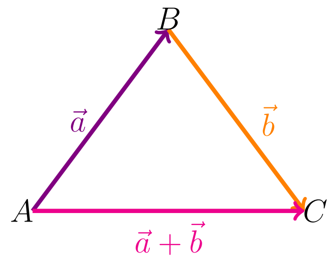
$$\begin{aligned} a < 0 \implies |a| = -a \wedge b > 0 \implies |b| = b & \quad \text{by definition of absolute value} \qquad (1.5) \\ \implies a \cdot b < 0 & \quad \text{since } a \text{ is negative and } b \text{ is positive} \qquad (1.6) \\ \iff |a \cdot b| = -a \cdot b & \quad \text{by definition of absolute value} \qquad (1.7) \\ \implies |a \cdot b| = |a||b| & \quad \text{by (5)} \qquad (1.8) \end{aligned}$$

For 4

$$\begin{aligned} a < 0 \implies |a| = -a \wedge b < 0 \implies |b| = -b & \quad \text{by definition of absolute value} \qquad (1.9) \\ \implies a \cdot b > 0 & \quad \text{since } a \text{ and } b \text{ are negative} \qquad (1.10) \\ \iff |a \cdot b| = a \cdot b & \quad \text{by definition of absolute value} \qquad (1.11) \\ \implies |a \cdot b| = |a||b| & \quad \text{by (9)} \qquad (1.12) \end{aligned}$$

$\square$

Proposition. $|a+b| \leq |a| + |b|$
The Triangle Inequality is arguably the most important inequality in all of analysis and later on within Metric Spaces. Graphically this inequality would look something like this



Proof. Given that the absolute value involves cases then the proof to shall follow by cases and thus involves four cases where $a, b \neq 0$ as this would give the trivial result.

1. $a > 0, b > 0$
2. $a > 0, b < 0$
3. $a < 0, b > 0$
4. $a < 0, b < 0$

For 1.

$$a > 0 \implies |a| = a \wedge b > 0 \implies |b| = b \quad \text{by definition of absolute value} \tag{1.13}$$
$$a + b > 0 \quad \text{since } a \text{ and } b \text{ are positive} \tag{1.14}$$
$$\implies |a+b| = a + b \quad \text{by definition of absolute value} \tag{1.15}$$
$$\implies |a+b| = |a| + |b| \quad \text{by (13)} \tag{1.16}$$

For 2. and similarly 3. This case leads to two further subcases depending on whether b is larger than a.

$$|a+b| \leq a - b$$

$$i)\ a + b \geq 0 \qquad ii)\ a + b \leq 0 \tag{1.17}$$
$$P.f\ \ a + b \leq a - b \qquad a + b \leq a - b \tag{1.18}$$
$$\implies b \leq -b \qquad \implies a \leq -a \tag{1.19}$$
$$\text{True } \because b \leq 0 \wedge -b \geq 0 \qquad \text{True } \because a \geq 0 \wedge -a \leq 0 \tag{1.20}$$

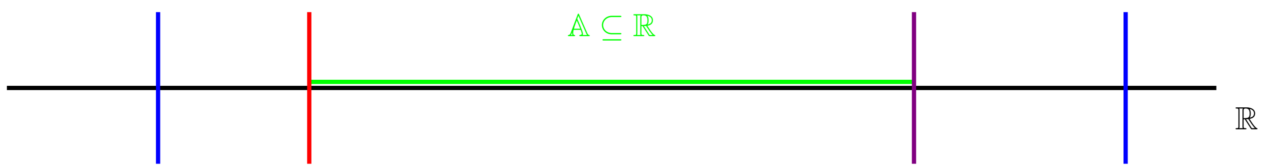
For 4

$$a < 0 \implies |a| = -a \wedge b < 0 \implies |b| = -b \quad \text{by definition of absolute value} \tag{1.21}$$
$$\implies a + b > 0 \quad \text{since } a \text{ and } b \text{ are negative} \tag{1.22}$$
$$\iff |a \cdot b| = a + b \quad \text{by definition of absolute value} \tag{1.23}$$
$$\implies |a+b| = |a| + |b| \quad \text{by (21)} \tag{1.24}$$

□

1.1.3 Suprema and Infima

A completely ordered field is one which also satisfies the Completeness Axiom which will be defined after introducing the notion of Suprema and Infima.



Consider the diagram above where in green we find a subset of $\mathbb{R}$ now in violet and in blue to the right the two positions represent elements in $\mathbb{R}$ s.t.

$$\forall x \in A, x \leq z$$

Where z represents any element in $\mathbb{R}$ which is larger than the largest member of A. Such an element is known as an element z is known as an *upper bound* of A and the set A is known as *bounded above* if it posses an upper bound.
Both the blue (to the right) and violet elements are upper bounds of A but the violet element is evidently the *least upper bound of* A and this is referred to as the supremum of A, denoted $\sup(A)$.

$$\begin{aligned} \sup(A) : \alpha &= \sup(A) \\ \iff i)\ \alpha &\geq a \in A \\ \iff ii)\ \exists\ \beta \geq a \in A &\implies \alpha \leq \beta \end{aligned}$$

Very similarly the notion of an infinum may be defined.

Proposition. Given the existence of a supremum or infinum in a set, such a bound must be unique.

Proof.

The proof will follow by contradiction

$$\begin{aligned} \text{Assume } \exists S_1 \wedge S_2 \text{ suprema } \in S : S_1 &\neq S_2 \\ \implies \forall s \in S,\ S_1 \geq s \wedge S_2 &\geq s && \text{by def'n of Suprema} \\ \implies S_1 \geq S_2 \wedge S_2 &\geq S_1 \\ \iff S_1 &= S_2 \text{ ※} \end{aligned}$$

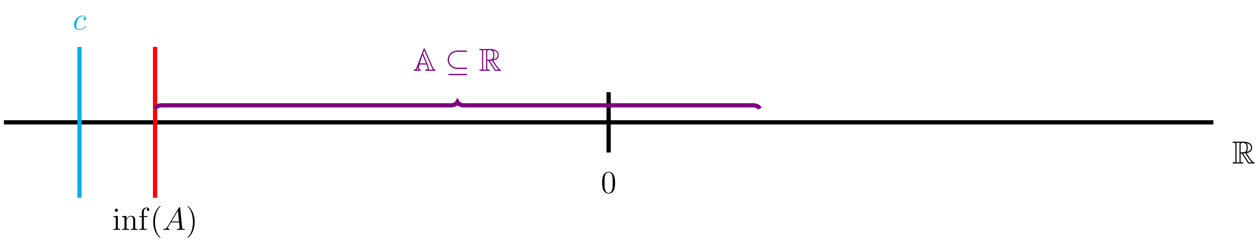
$\square$

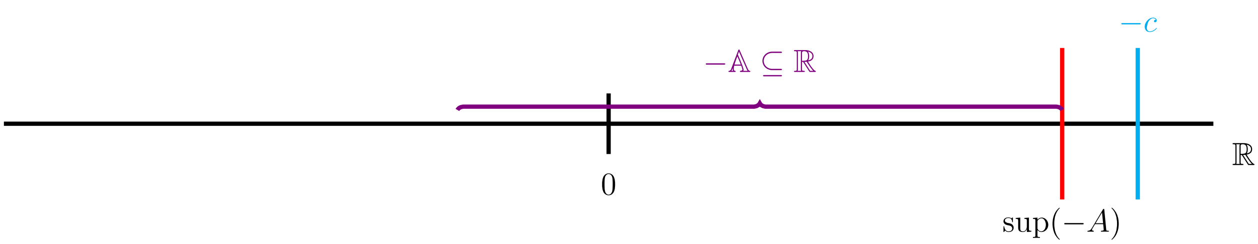
1.1.4 The Completeness Axiom

The completeness axiom defines complete a ordered field and is the nature of $\mathbb{R}$.Having defined Suprema the CA may be stated as follows; for any non-empty, bounded above subset of $\mathbb{R}$ there exists a real number which is the supremum of the said subset of $\mathbb{R}$.
This maybe restated to show the existence of an infinum for any bounded below subset of $\mathbb{R}$ by making use of

$$\inf(A) = -\sup(-A)$$

as *Every bounded below, non-empty subset of* $\mathbb{R}$ *has an infinum.*





Equivalence of Supremum CA and Infinum CA

Proof. These two statements of the completeness axiom are shown to be equivalent starting from the infinum statement

Consider $-A = \{-a : a \in A\}$ where A is bounded below

$$\begin{aligned}
\exists c \in \mathbb{R} : c \leq a \,\forall\, a \in A & \qquad \text{by def'n of lower bound} \\
\implies -a \leq -c \,\forall\, a \in A & \qquad \text{taking the negative} \\
\implies -A \text{ is bounded above} & \\
\because A \subseteq \mathbb{R} \implies \exists \sup(-A) = S & \qquad \text{by the completeness axiom}
\end{aligned}$$

It is clear from the figure that $-\sup(-A)$ is a lower bound for A

$$\begin{aligned}
\implies -S \leq a \forall a \in A & \quad \text{showing that } -\sup(-A) \text{ is a lower bound} \\
\exists c : c \leq a \forall a \in A & \\
\implies -a \leq -c & \qquad -c \text{ is an upper bound for -A}
\end{aligned}$$

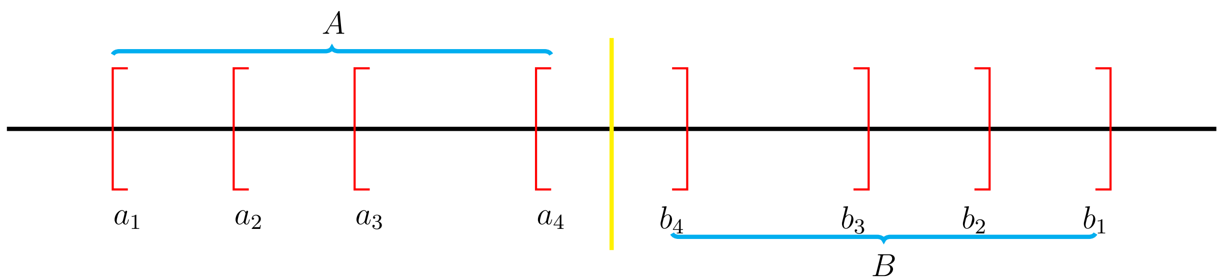
But S is the supremum

$$\implies S \leq -c \qquad -S \text{ is supremum}$$
$$\implies -S \geq c$$

$\therefore -S$ is the greatest lower bound, infinum.

$\square$

The Nested Interval Property



Theorem 1.1.1. *The Nested Interval Property states that; $\forall n \in \mathbb{N}$ assume we are given a closed interval $I_n = [a_n, b_n]$ such that this contains other closed nested intervals*

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_n$$

which have a non-empty intersection

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Proof. The proof follows as a result of the completeness axiom.
Consider all the nested intervals and think of one side of each interval, say the lower limit of each interval. This forms a set which will be denoted A
It is evident that each upper limit of every interval will act as an upper bound for every element the set A. By the **completeness axiom** this set has a **Supremum** say

$$s = \sup A$$

Now for containment we show that this supremum is found in each I_n by considering that by definition of the supremum

$$s \leq b_n \ \forall \ \in \mathbb{N}$$

Given that it is the least upper bound and B is the set of all upper bounds for A. But s is also an upper bound for A

$$a_n \leq s \leq b_n \ \forall \ \in \mathbb{N}$$
$$\therefore s \in \bigcap_{n=1}^{\infty} I_n$$

$\square$

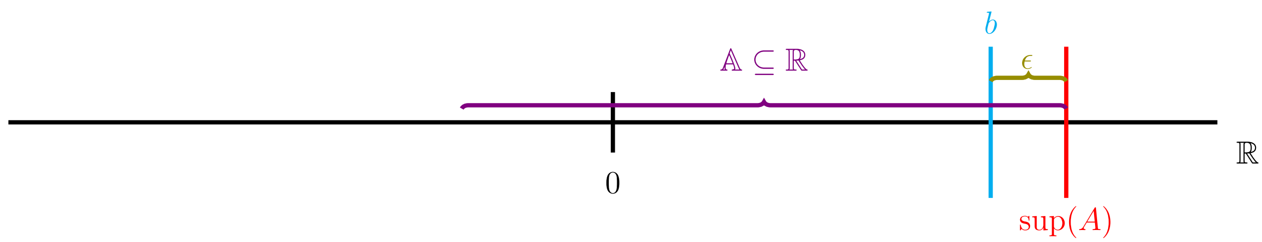
Maximum denoted $\max(A)$ for a set A is the largest element of a bounded above set and in other words the least upper bound for a bounded above set, the supremum.

1. $\max(A) \in A$
2. $\max(A) \geq a \, \forall a \in A$

Meaning that if a supremum exists then it is not necessarily the maximum as a supremum is not necessarily part of the set it pertains to .But if a maximum exists then it is also said to be the supremum of such a set. Similarly the **Minimum** can be defined for bounded below sets.

Alternate formulation of Suprema and Infima *Moving just to left of the supremum or just to the right of the infinum one will always find themselves in the set which the supremum or infinum pertain to.* Let $A \subseteq \mathbb{R}$ have an upper bound $s \in R$

$$\boxed{\implies s = \sup(A) \iff \forall \epsilon > 0 \, \exists \, a \in A : s - \epsilon < a}$$



Proof. Given that this is a bi-conditional statement the proof will follow by cases

Proving the ($\implies$) case, directly.

$$\begin{aligned} \text{Let } s = \sup(A) \wedge \epsilon > 0 & \\ \text{Consider } s - \epsilon < s & \qquad \text{by order} \end{aligned}$$

$\because$ s is the least upper bound $\implies$ $s - \epsilon$ is not an upper bound

$$\implies \exists a \in A : s - \epsilon < a$$

Proving the ($\impliedby$) case, by contradiction

$$\begin{aligned} \text{Let } \forall \, \epsilon > 0 \exists a \in A : s - \epsilon < a & \\ \text{Suppose } s \neq \sup(A) & \\ \implies \exists b : b \geq a \, \forall \, a \in A \wedge b < s & \qquad \text{by supposition} \\ \text{Let } \epsilon = s - b & \\ \implies \exists \, a \in A : s - \epsilon < a & \qquad \text{by the premise} \\ \implies s - s + b < a \text{ ※} & \qquad b \text{ is an upper bound} \end{aligned}$$

$\square$

1.1.5 The Density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

The notion of density stems from the definition *A subset X of R is said to be a "dense" subset of* $\mathbb{R}$ *if for every* $w \in \mathbb{R}$*,there is a sequence* $\{x_n\}$ *of numbers in X which converges to* w. This definition is beyond the scope of this course and we are to think of it as follows; say $\exists W \subseteq \mathbb{R}$ then W s.t.b dense in $\mathbb{R}$ iff $\forall x, y \in \mathbb{R} \wedge w \in W, x < w < y$.
This is the definition to be employed here in and is such that between every two real numbers another number which is an element of a dense subset of $\mathbb{R}$ can be found between the two said real numbers.
As seen above $\mathbb{Q}$ and $\mathbb{I}$ are **dense subsets** of $\mathbb{R}$. This shall be proved below.

Theorem: The Archimedean Property of $\mathbb{R}$
Another property of the real numbers which shall be employed to show the density of the field is the Archimedean Property and can be stated as such *The set of natural numbers has no upper bound*. The statement can be equivalently presented as the following within the context of the real numbers

1. $\forall a \in \mathbb{R}, \exists n \in \mathbb{N} : n > a$ There is no largest real number
2. $\forall x, y \in \mathbb{R}, x > 0, \ \exists \ n \in \mathbb{N} : nx > y$ There is no smallest real number
3. $\forall x \in \mathbb{R}, x > 0 \ \exists \ n \in \mathbb{N} : 0 < \frac{1}{n} < x$ There is no smallest real number

1.*Proof.* To prove that $\mathbb{N}$ is not bounded above an approach by contradiction is taken. Suppose that $\mathbb{N}$ is bounded above

$$\begin{aligned} &\implies \exists \alpha : \forall n \in \mathbb{N}, \ \alpha \geq n && \because \ \mathbb{N} \neq \emptyset \\ &\implies \alpha \geq n + 1 \ \forall n \in \mathbb{N} && \text{since } \alpha \text{ is the least upper bound} \\ &\implies \alpha - 1 \geq n \ \forall n \in \mathbb{N} ※ \end{aligned}$$

contradicting the fact that α is the least upper bound.Thereby proving that $\mathbb{N}$ is not bounded above. □

2.*Proof.* To prove the second equivalent statement of the Archimedean Property the proof above is used as a lemma towards contradiction

$$\begin{aligned} \text{Consider } nx > y = n > \frac{y}{x} && \because x > 0 \\ \text{Suppose } \nexists \ n > \frac{y}{x} && \text{premise of contradiction} \\ \implies n < \frac{y}{x} \ \forall \ n \in \mathbb{N} \end{aligned}$$

This states that $\frac{y}{x}$ is an upper bound for $\mathbb{N}$ contradicting the lemma.Thus the second statement is shown to be true. □

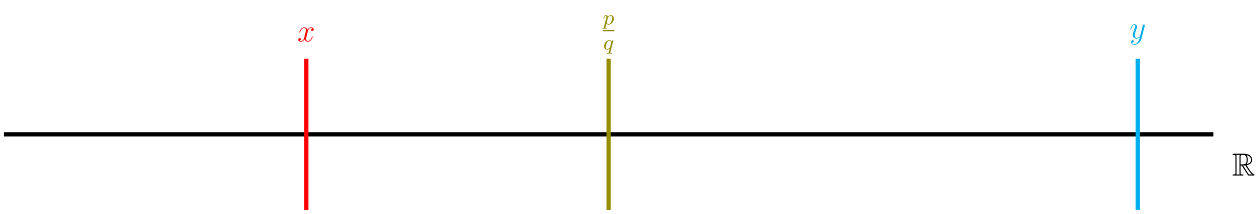
3.*Proof.* To prove the third equivalent statement of the Archimedean Property the first above is used again as a lemma towards contradiction

$$\begin{aligned} \text{Suppose } \frac{1}{n} &\geq \epsilon : \epsilon \in \mathbb{R} && \text{premise of contradiction} \\ \implies n &\leq \frac{1}{\epsilon} \ \forall n \in \mathbb{N} \end{aligned}$$

This states that $\frac{1}{\epsilon}$ is an upper bound for $\mathbb{N}$ contradicting the lemma.Thus the third statement is shown to be true.

Theorem: The Density of $\mathbb{Q}$ in $\mathbb{R}$

$$x, y \in \mathbb{R} : x < y \ \exists \ q \in \mathbb{Q} : x < q < y$$



Proof. Consider that it is known that rational numbers are of the form $\frac{p}{q} : p, q \in \mathbb{R}$.With this notion in mind we can establish the proof by figuring out what these p and q would be to exist within a continuous interval of two real numbers x and y.

$$\begin{aligned} \text{Let } x, y \in \mathbb{R} : x < y & \\ \implies \frac{1}{y-x} \in \mathbb{R} & && \because y - x > 0 \\ \exists N \in \mathbb{N} : N > \frac{1}{y-x} : y - x > \frac{1}{N} & && \text{By the Archimedean Property (2)} \\ \text{Consider } A \subseteq \mathbb{Q} : A = \{\frac{m}{N} : m \in \mathbb{N}\} & \end{aligned}$$

Claim: $A \cap (x, y) \neq \emptyset$ implying that there is a rational number between the two real numbers.Proceeding by contradiction

$$\implies A \cap (x, y) = \emptyset \qquad \text{Premise of contradiction}$$

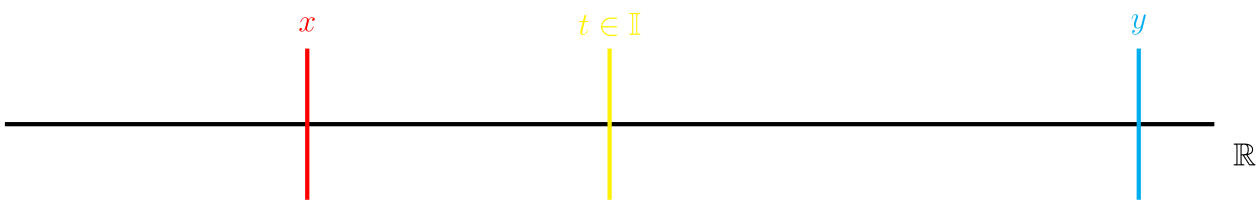
Taking m_1 such that m_1 is the greatest integer satisfying this inequality $\frac{m_1}{N} < x$

$$\begin{aligned} \implies \frac{m_1 + 1}{N} &> y && \text{Given that } x - y > \frac{1}{N} \\ \implies x - y < \frac{m_1 + 1}{N} - \frac{m_0}{N} &= \frac{1}{N} \\ \implies \frac{1}{N} &< x - y \text{ ※} \end{aligned}$$

$\square$

Theorem: The Density of $\mathbb{I}$ in $\mathbb{R}$

$$x, y \in \mathbb{R} : x < y \; \exists \; t \in \mathbb{I} : x < t < y$$



Proof. Making use of the density of rational numbers in $\mathbb{R}$ and the knowledge that $\sqrt{2} \in \mathbb{I}$ it will be shown that an irrational number must exist between two real numbers x, y.

$$\begin{aligned}
\text{Let } x, y \in \mathbb{R} : x < y & \\
\implies \frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}} & \\
\exists r \in \mathbb{Q} : \frac{x}{\sqrt{2}} < \sqrt{2}r < \frac{y}{\sqrt{2}} & \qquad \text{By the density of of } \mathbb{Q} \text{ in } \mathbb{R} \\
\sqrt{2}r = \frac{p}{q} : p, q \in \mathbb{N} & \qquad \sqrt{2}r \in \mathbb{Q} \text{ by density} \\
\text{But } r = \frac{m}{n} : m, n \in \mathbb{N} & \qquad r \in \mathbb{Q} \\
\implies \sqrt{2}\frac{m}{n} = \frac{p}{q} & \\
\implies \sqrt{2} = \frac{pm}{nq} \in \mathbb{Q} ※ & \\
\therefore \sqrt{2} \in \mathbb{I} \implies \sqrt{2}r \in \mathbb{I} & \\
\therefore \sqrt{2}r \in (x, y) & \qquad \text{by contradiction}
\end{aligned}$$

□

1.1.6 An introduction to $\mathbb{C}$

Some definitions for $\mathbb{C}$ Any element in the complex set is defined as an ordered pair of real numbers

$$z = (a, b) \in \mathbb{R} \times \mathbb{R} : a + ib \wedge i^2 = -1$$

a is referred to as the real part of z whilst b is the imaginary part. The complex numbers form a field under addition and multiplication just as the real numbers did and thus follow the 11 rules stated in **section 1.1**. Of course addition and multiplication work a bit differently for complex numbers and that's what will be outlined within this section. Consider $z_1, z_2 \in \mathbb{C} : z_1 = (a, b), z_2 = (c, d)$

Addition	$(a,b)+(c,d)=(a+c,b+d)$
Multiplication	$(a,b)\cdot(c,d)=(a\cdot c-b\cdot c, a\cdot d+b\cdot c)$

Multiplicative Inverse in $\mathbb{C}$ To derive this notion let us consider its definition and that of the **identity** where a multiplicative inverse would take us to the identity

$$(a,b)\cdot(x,y)=(1,0)$$

Applying the definition of multiplication achieved earlier two simultaneous equations are attained

$$\begin{cases} ax-by=1 \\ bx+ay=0 \end{cases}$$

Solving this one achieves

$$(a,b)\cdot\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(1,0)$$

Which defines the multiplicative inverse for any complex number (a,b).

Some Facts in $\mathbb{C}$

Theorem 1.1.2. *Let z and w be complex numbers. Then*

1. $\bar{\bar{z}}=z$
2. $\bar{z}=z \iff z:z=(a,0)\,\exists\, a\in\mathbb{R}$
3. $\overline{z+w}=\bar{z}+\bar{w}$
4. $\overline{-z}=-\bar{z}$
5. $\overline{z\cdot w}=\bar{z}\cdot\bar{w}$
6. $\overline{z^{-1}}=\bar{z}^{-1}$, if $z\neq 0$
7. $|z|^2=z\cdot\bar{z}$
8. $|z\cdot w|=|z|\cdot|w|$
9. $|z+w|\leq|z|+|w|$

These statements are all trivially proven barring (9) **The Triangle Inequality in $\mathbb{C}$** which will be proven here under directly.

Proof. The proof will follow directly by cases.

1. $z=0 \vee w=0$
2. $z=\lambda w\,\forall\,\lambda\in\mathbb{R}:\lambda>0$

3. $z = \lambda w \,\forall\, \lambda \in \mathbb{R} : \lambda < 0$
4. $z \neq \lambda w \,\forall\, \lambda \in \mathbb{R} \wedge w \neq 0$

For 1. It is evident that $|z + w| \leq |z| + |w|$ holds by cases of equality

$$\begin{aligned} |0 + w| &= |0| + |w| \\ |w| &= |w| \end{aligned} \qquad \begin{aligned} |z + 0| &= |z| + |0| \\ |z| &= |z| \end{aligned} \qquad \begin{aligned} |0 + 0| &= |0| + |0| \\ 0 &= 0 \end{aligned}$$

For 2. Consider that by the case $w = \frac{1}{\lambda} z$

$$\begin{aligned} &\left| z + \frac{1}{\lambda} z \right| && \text{by premise of case} \\ &z + \frac{1}{\lambda} z && \text{by definition of absolute value} \\ &|z| + \left| \frac{1}{\lambda} z \right| && \text{by definition of absolute value} \\ \implies &\left| z + \frac{1}{\lambda} z \right| = |z| + \left| \frac{1}{\lambda} z \right| \end{aligned}$$

Meaning that this case is also true by equality and since for case 3 $z > \frac{1}{\lambda}$ then similarly 3 holds also.

For 4. It is evident that the following quadratic equation is sensible

$$\begin{aligned} 0 &< |z - \lambda w|^2 \\ 0 &< (z - \lambda w) \cdot \overline{(z - \lambda w)} && \text{by fact 7} \\ 0 &< (z - \lambda w) \cdot (\bar{z} - \lambda \bar{w}) && \text{by facts 4 \& 5} \\ 0 &< z\bar{z} + \lambda^2 w\bar{w} - \lambda(w\bar{z} + z\bar{w}) && \text{by def'n of multiplication in } \mathbb{C} \\ 0 &< |z|^2 + \lambda^2 |w|^2 - \lambda(w\bar{z} + z\bar{w}) && \text{by fact 7} \end{aligned}$$

It is clear that we've now got a quadratic equation in terms λ on our hands with a discriminant of the form

$$(w\bar{z} + z\bar{w})^2 - 4|w|^2 \cdot |z|^2 < 0 \qquad \text{Real Coefficients with no Real Sol.}$$

Since $(w\bar{z} + z\bar{w}), |w| \cdot |z| \in \mathbb{R}$ and also $|w| \cdot |z| \geq 0$

$$|w\bar{z} + z\bar{w}| < 2|w| \cdot |z| \qquad \psi$$

Now consider $|z + w|^2$ with the form of triangle inequality in mind

$$\begin{aligned} |z + w|^2 &= (z + w) \cdot (\bar{z} + \bar{w}) && \text{by fact 7} \\ |z + w|^2 &= |z|^2 + |w|^2 + (w\bar{z} + z\bar{w}) \\ |z + w|^2 &< |z|^2 + |w|^2 + 2|w| \cdot |z| && \text{by the inequality } \psi \\ |z + w|^2 &< (|z| + |w|)^2 \\ \implies |z + w| &< |z| + |w| \end{aligned}$$

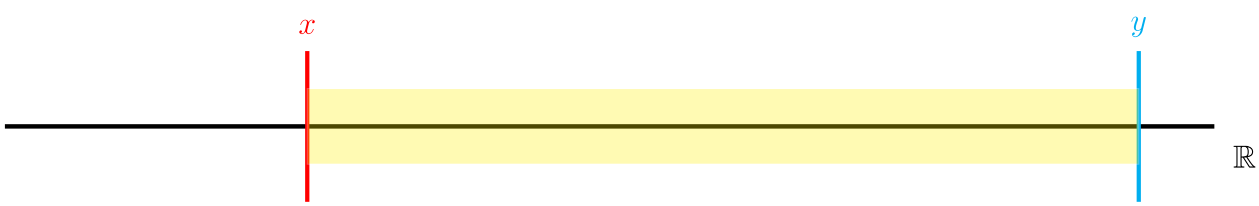
$\square$

1.1.7 Intervals

Within analysis it is often or perhaps even the main point of interest to understand what is happening within some function for a given distance, a region of sorts around a point. The nature of such regions is fundamentally different for functions which exist in $\mathbb{C}$ or $\mathbb{R}$.

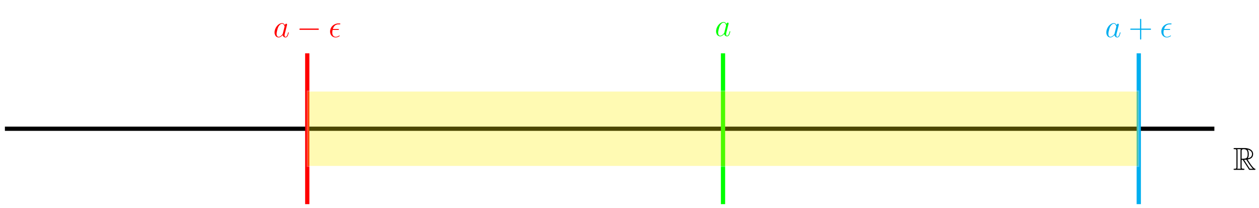
Intervals are typically presented in one of two ways.
Either describing some region **between** two points a and b as $|a-b|$ which can be denoted (a,b) or $]a,b[$ if the interval is **open** meaning that the end points are not included and a point in the is interval would be of the form

$$x \in \{x : a < x < b\}$$



Or alternatively describing the interval as an area **around** some point where x can lie near this point within a given limit or range typically denoted ϵ. Again this distance is **open** or **closed** depending on the inclusion of the extremities.

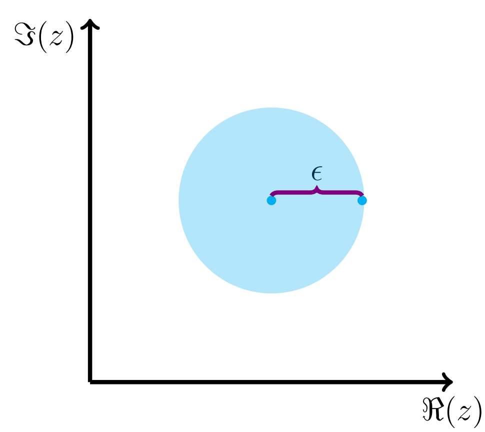
$$|x-a| < \epsilon \implies a-\epsilon < x < a+\epsilon$$



Open Discs and Intervals in $\mathbb{C}$ Given that the complex plane is not one dimensional and involves things of the form

$$(a,b) \in \mathbb{R} \times \mathbb{R} : a+ib \wedge i^2 = -1$$

then when talking about distance from a point we can no longer thing about distance in a singular direction as on a one dimensional line but now within the context of a 2-D axis typically referred to as an **Argand Diagram** meaning we're going to start getting circles !



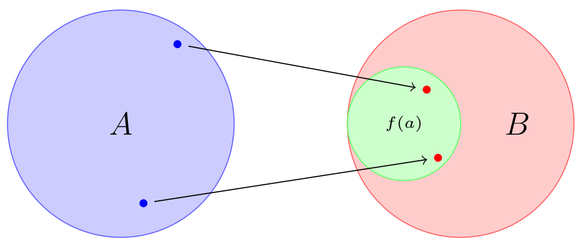
1.2 Functions & Countability

The size of sets or rather comparing the size of sets is
of great mathematical relevance both within the context of combinatorial mathematics but also within the context of analysis.
To compare the size of sets, functions are used thus below a brief foray into functions is given after which finiteness and countability will be discussed.

A **function** is a set relation from a set A to a set B which to satisfy a rigorous definition must satisfy two conditions. Using the notation $f : A \to B$ we indicate that

$$i)\ \forall a \in A\ \exists b \in B : (a, b) \in f$$

meaning that each element in A is assigned an element from B of the form $f(a)$ and the region $f(a)$ is known as the image of a under f also referred to as the function. The domain of the function is A and the target or codomain is B.



$$f : A \to B$$

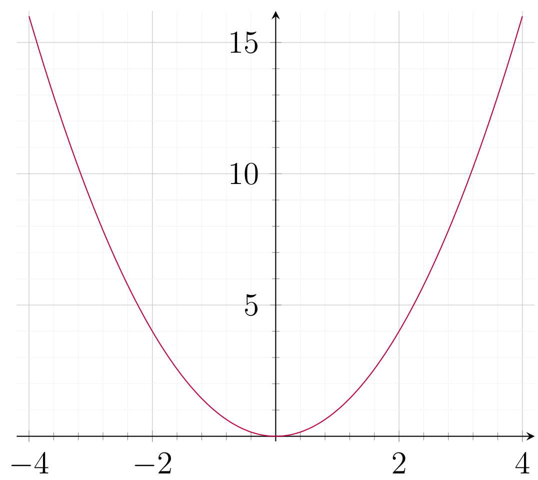
$$ii) \forall a \in A \wedge \forall b \in B,\ \text{if } (a, b) \in f \wedge (a, c) \in f \implies b = c$$

The second condition shown above is what is meant when one posits that a function is well-defined. For this condition to be satisfied to each element of A precisely one element of B is assigned. Thus each element of A is related to not more than one element of B.This infers that one-to-many mappings are not functions.

Given that a function is a set relation between to sets it can be viewed as a subset of the cartesian product of the domain and the codomain. Within this same light one can think of it as enforcing a rule ex: $y = x^2$ enforcing the rule that two variables are related by one's square.

$$\implies f : A \to B \subseteq A \times B$$

The graph of a function $f : A \to B$ is $\{(x, f(x)) : x \in A\}$ this is the pictorial representation of this rule that we are all familiar with, for this particular example of $y = x^2$ we know it would like something like this



For a function $f : X \to Y$ the image of T under f, $f(T)$ such that $T \subseteq X$ is the set

$$f(T) = \{y \in Y : \exists x \in T : f(x) = y\}$$

The **inverse image** of a function is denoted by f^{-1} and is not to be confused with the inverse function. The inverse image applies the same notion of the image but applying it to a subset of the codomain of the function. Considering $C \subseteq Y$ the set will look something like this

$$f^{-1}(C) = \{x \in X : \exists y \in C : f(x) = y\}$$

1.2.1 Surjectivity & Injectivity

The section title includes two fancy names which describe in what way the domain is being mapped to the codomain and so what type of range the function creates.

A One-to-One function is one for which

$$\text{if } f(a_1) = f(a_2) \text{ then } a_1 = a_2$$

meaning that as the name implies there is a one to one assignment between the domain elements and the related image elements. These of functions are also known as injective.In terms of cardinality, for a function $f : A \to B$, if such a function is injective then this must imply that

$$|A| \leq |B|$$

and this makes intuitive sense as if every element of A corresponds to an element in the image which is a subset of B, then the totality of B is larger than A

An on-to function is one for which the range is equal to the codomain so for

$$f : A \to B \implies ran(f) = B$$
$$\forall y \in B\, \exists\, x \in A : f(x) = y$$

This implies that every element of the domain is related to an element of the codomain. These of functions are also known as surjective.In terms of cardinality, for a function $f : A \to B$, if such a function is surjective then this must imply that

$$|A| \geq |B|$$

and this makes intuitive sense because if each element of A is mapped to every element in B then it stands to reason that it is either a many to one or a one to one correspondence.

It is good to appreciate that an injective relation is a forward one going from the domain to the image and can be approached as such whilst a surjective relation is a backwards kind of thing where one starts out talk about the range and moves back to the domain.

1.2.2 Bijections

A bijective function is one which is both injective and surjective. Informally this may be referred to as a one to one correspondance between sets and can be expressed

$$\forall b \in B \ \exists \text{ precisely one } a \in A : f(a) = b$$

This infers that every element in the domain corresponds to single element in the range.

If one is able to form a bijection between two sets then it becomes clear that these two sets possess the same cardinality or size.

$$|A| = |B|$$

1.2.3 Compositions

Consider two functions f and g

$$f : A \to B \qquad\qquad g : C \to D$$

Their composition from A to D is the function $g \circ f : A \to D \equiv g(f(x))$

The composition of functions is associative but not commutative.

$$h \circ (g \circ f) = (h \circ g) \circ f \qquad\qquad f \circ g \neq g \circ f$$

Proof: **Associative Law :-** $h \circ (g \circ f) = (h \circ g) \circ f$

$$\begin{aligned}
\text{Restating } f : A \to B, g : B \to C \text{ and } h : C \to D &\implies h \circ (g \circ f) = (h \circ g) \circ f \\
\text{Consider } g \circ f : A \to C &\implies h \circ (g \circ f) : A \to D \\
\text{Consider } h \circ g : B \to D &\implies (h \circ g) \circ f : A \to D
\end{aligned}$$

$$\begin{aligned}
\text{Let } x \in (g \circ f) \circ h & \qquad \text{Let } x \in (h \circ g) \circ f \\
\iff g(f(x)) \circ \text{ and } h(g(x)) \circ f & \qquad \text{by def'n of functions} \\
\therefore h(g(f(x))) = h(g(f(x))) & \qquad \text{by def'n of functions } \square
\end{aligned}$$

1.2.4 Inverse Functions

A bijective function $f : A \to B$ that means that there exists a well-defined function $g : B \to A$ which undoes the effect of f. Now their compositions $g \circ f : A \to A = i_A$ and $f \circ g : B \to B = i_B$. The function g which implies these properties is known as the inverse function of f denoted by f^{-1}. It is well-defined for bijective functions itself being bijective.
Consider two functions f and g

$$f : A \to B \qquad\qquad g : C \to D$$

This implies that $g \circ f : A \to D$ if it is a bijective function has an inverse $f \circ g : D \to A$.

1.2.5 Character Persistence in Composition

If $f : A \to B$ and $g : B \to C$ are injective functions show that $g \circ f$ is an injective function also.

$$\begin{aligned}
\text{Consider } g \circ f : A \to C & \qquad \text{by definition of composition} \\
\text{Let } x_1, x_2 \in A : g \circ f(x_1) = g \circ f(x_2) & \\
\iff g(f(x_1)) = g(f(x_2)) & \qquad \text{by definition of composition} \\
\text{Let } y_1 = f(x_1), y_2 = f(x_2) & \\
\implies g(y_1) = g(y_2) & \\
\iff y_1 = y_2 & \qquad \text{by the injectivity of g} \\
\implies f(x_1) = f(x_2) & \\
\iff x_1 = x_2 & \qquad \text{by the injectivity of f}
\end{aligned}$$

$$\therefore g \circ f \text{ is an injective function } \square$$

The converse of this proof would be that if a composite function is injective then the functions comprising it must be injective also. A counter example to this would be the function $f \circ g(x) = e^{2x}$ which is the composite function of $f(x) = x^2$ a non-injective function and $g(x) = e^x$ an injective function.

If $f : A \to B$ and $g : B \to C$ are surjective functions show that $g \circ f$ is a surjective function also.

$$\begin{aligned}
\text{Consider } g \circ f : A \to C & \qquad \text{by definition of composition} \\
\text{Let } z \in C & \\
\iff y \in B, \exists z : g(y) = z & \qquad \text{by surjectivity of g} \\
\iff x \in A, \exists y : f(y) = x & \qquad \text{by surjectivity of f} \\
\implies x \in A, \exists z \ : g(f(y)) = z & \\
\iff x \in A, \exists z \ : g \circ f(x) = z & \qquad \text{by definition of composition}
\end{aligned}$$

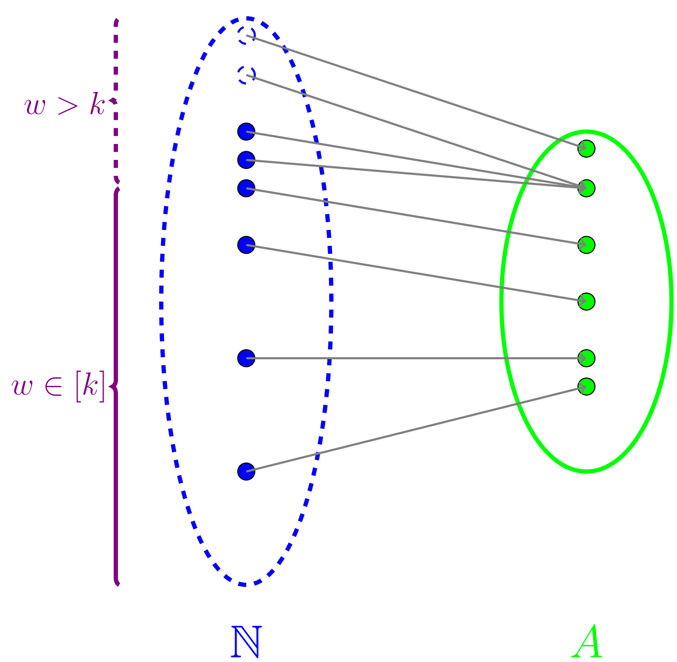
$$\therefore g \circ f \text{ is surjective also } \square$$

1.2.6 Countable Sets

A set A is said to be *countable* if a bijection can be formed from this set onto some subset of $\mathbb{N}$,$\mathbb{N}_n$, sharing the same cardinality n, of A. This is the case for all finite sets.
Otherwise it is known as *uncountable*.

Tweaking the orange box definition some what we can use a surjection to show a many to one relation between $\mathbb{N}$ and a set A rather than a bijection with a subset of $\mathbb{N}$. The resulting statement has the same effect but has no need to make use of the notion a subset of $\mathbb{N}$.
Theorem 1.2.1. *A non-empty set is countable* $\iff$ $\exists\ f : \mathbb{N} \to A$ *which is surjective*

Proof. Surjectivity in terms of cardinality infers that $\mathbb{N}$ is larger than A.
($\implies$)This gives two cases following the premise that A is countable;
The first case is relatively trivial for the case where A is **countably infinite**. This means it shares the same cardinality as $\mathbb{N}$ and $f : \mathbb{N} \to A$ is bijective and as a result surjective as required.
Now considering the case where A is **countably finite**.



In the figure above the dashed lines show the infinite nature of the set $\mathbb{N}$. The figure shows that assuming that there is a bijection for some section of the two sets, given that there is an infinite number of elements in $\mathbb{N}$ they surely exhaust all the elements of A.

$$\begin{aligned}
&\exists\, f : \mathbb{N}_w \to A && \\
\implies\; & n_w = f(n_w)\,\forall\, w \in [k] && \text{Bijective subsection} \\
& n_{w>k} = f(n_k)\ \exists && \\
& n_{w>k} \in \mathbb{N} && \text{many-to-one}
\end{aligned}$$

If there is some unused element in A by this many to one relation then given that $\mathbb{N}$ is infinite this element will to be used within the vain of the Hilbert Hotel case. The whole of A is exhausted by the infinite nature of $\mathbb{N}$ under the supposition the A is finite.

Now considering ($\impliedby$) we start with the presumption that the function is surjective to show that A is countable

Considering $f : \mathbb{N}_w \to A$ is surjective

$$\implies \forall\, a \in A\ \exists\, n \in \mathbb{N} : f(n) = a$$

Now consider $f^{-1}(a)$ which would take each element in A to some element in a **subset** of $\mathbb{N}$

$$\implies f^{-1}(a_1) = f^{-1}(a_2) \implies a = b$$

Thus f^{-1} is injective meaning that there is a **bijection** between A and some subset of N.

Therefore A is countable by definition.

□

The following are stated without proof

- If a bijection can be formed from a set A to $\mathbb{N}$ then such a set is said to be *countably infinite.*
- $|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B|$ - Cantor-Bernstein Theorem

Corollary 1.2.1.1. *If A is countable and $f : A \to B$ is surjective then B is countable.*

Proof. The proof will follow directly from the definition of countability and surjectivity as well as the inverse function technique made use of in the preceeding theorem.

$$\begin{aligned}
\implies \exists\, \psi : W \to A : W \subseteq \mathbb{N} \text{ which is bijective} && \text{By def'n of countability} \\
f : A \to B \text{ is surjective} && \text{by premise} \\
\implies |A| \geq |B| && \\
\implies \exists\, Z \subseteq A : f^{-1} : B \to Z \text{ is bijective} && \text{by surjectivity} \\
\implies f \circ \psi \text{ is bijective} && \text{Persistance of composition} \\
\implies \exists Q : \mathbb{N} \to B \text{ is surjective} &&
\end{aligned}$$

Therefore B is countable also

□

Theorem 1.2.2. $\mathbb{N} \times \mathbb{N}$ *is countably infinite*

Proof. We are to construct a pairing function that maps bijectively from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ to show the countability. A multitude of such functions can be expressed as shown by Cantor but the function used hereunder is given as it is the "textbook" example.
Consider the function $f(n, m) = 2^n 3^m$, for proof it will be showing to be bijective.

$$\text{For injectivity: } f(p, q) = f(n, m) \implies (p, q) = (n, m) \text{ to be shown}$$
$$2^p 3^q = 2^n 3^m$$

This infers that the image or codomain is composed of products of the primes, 2 and 3

$$\implies p = n \wedge q = m \text{ By the fundamental theorem of arithmetic}$$

For surjectivity, one considers that the pairs have the cardinality of the range and as such the domain must share the cardinality of the range. This implies that the range and image are equal and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is surjective.

The function is thus bijective and $\mathbb{N} \times \mathbb{N}$ is countable by definition

□

Theorem 1.2.3. *If A_n is countable $\forall n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.*

Proof. The proof will follow by showing that there is a bijection with some subset of $\mathbb{N}$ or some other set we have shown to be countable above.
A union of sets is quite messy and difficult to handle given that it is a set of many elements from an infinite number of sets so it stands to reason that organising these elements will be productive. An array is constructed where the ith row contains all the elements of the set A_i and the jth column represents the jth elements from the A_n sets.

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & \ddots & \ldots & \ldots & \ldots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{matrix}$$

Now it is evident that given $i, j \in \mathbb{N}$ a function may be constructed which given an i, j entry gives an element in the set. Such a function would surely be of the form $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (of the form as the function is actually $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n$) which was shown to be bijective above and thus by the countability of $\mathbb{N} \times \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} A_n$ is countable also. □

Note that this argument is obviously extendible to finite unions of countable sets by considering subsets of $\mathbb{N}$.

Theorem 1.2.4. $\mathbb{Q}$ *is countable*

Proof. The proof will follow by recalling that $\forall\ q \in \mathbb{Q}^+\ \exists \frac{n}{m} = q : n, m \in \mathbb{N}$ thusly a pairing function may be constructed in such a way where (n, m) correspond to $\frac{n}{m}$ and so $f(n, m) = q$. This pair is clearly constructed from $\mathbb{N} \times \mathbb{N}$ and so the pairing function would be of the form

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$$

This function is clearly bijective given that each pair is related to precisely one q in an exhaustive manner by definition of $q \in \mathbb{Q}^+$ meaning that $\mathbb{Q}^+$ is **countably infinite** by theorem 1.2.2. A similar function can be be constructed for $\mathbb{Q}^-$ and since

$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$$

Then by **Theorem 1.2.3.** $\mathbb{Q}$ is countable. □

Theorem 1.2.5. *The finite product of countable sets is countable*

Proof. The proof will follow by induction.
Consider some family of countable sets $A_{[n]}$.For the **base case**, the cartesian product of two countable sets A_1 and A_2 is considered.Given that they are both countable then both of them may form surjections with $\mathbb{N}$

$$f : \mathbb{N} \to A_1 \qquad\qquad g : \mathbb{N} \to A_2$$

This infers that a surjection using these functions together is attainable in the following way

$$h : \mathbb{N} \times \mathbb{N} \to A_1 \times A_2$$
$$\implies h : (a, b) \to (f(a), g(b))$$

By **Theorem 1.21** and **Theorem 1.22** $A_1 \times A_2$ is countably infinite given that it forms a surjection with $\mathbb{N} \times \mathbb{N}$.
Now the **inductive hypothesis** will be to assume that some subset $A_{[k]}$ of the family $A_{[n]}$ is countable and now for proof it will be shown that A_{k+1} is countable also.
A function ψ may be constructed pairing the product set $A_{[k]}$, which is countable by the induction hypothesis to the set A_{k+1}, which is countable by the premise.

$$\psi : (A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1} \to (A_1 \times A_2 \times \cdots \times A_k \times A_{k+1}) = A_{[k+1]}$$

$$A_{[k]} \times A_{k+1} \to A_{[k+1]}$$

Now consider that in the **domain** of the function, the pairing side. As mentioned above, the product set is countable by the induction hypothesis and the set A_{k+1} is countable by the premise. By applying the same reasoning used to show that the base case stands it becomes evident that the domain is **countable**.
Evidently this function is bijective and thus $A_{[k+1]}$ is **countable**. □

1.2.7 Uncountable Sets

Theorem 1.2.6. $\mathbb{R}$ *is uncountable*

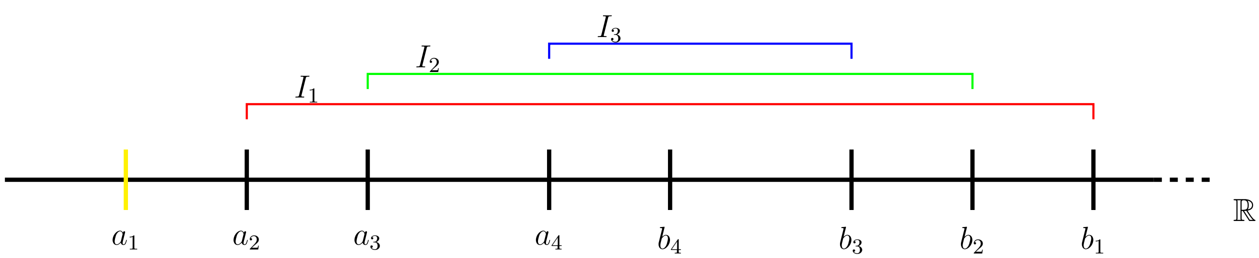
Proof. The proof will proceed by contradiction.
Recalling the definition of countability, for contradiction, it will be assumed that

$$\exists\ f : \mathbb{N} \to \mathbb{R},\ \forall x \in \mathbb{R}\ \exists\ n \in \mathbb{N} : f(n) = x$$

Where the condition at the end of the statement is surjectivity. Given that $\mathbb{R}$ is infinite it can also be said that this function would be injective and so bijective by their common infinite nature.
This would suggest that every element in $\mathbb{R}$ can be **labelled**

$$\mathbb{R} = \{x_1, x_2, \ldots, x_n, \ldots | \forall n \in \mathbb{N}\}$$

Now breaking this set down into nested intervals such that an interval $I_n = [a_{n+1}, b_n]$ does not contain the element n and contains the interval I_{n+1} we visualise



Now considering the intersection of these closed intervals, it is obvious that

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

By the premise that $I_n = [a_{n+1}, b_n]$ and as visualised by the point a_1.
Recalling that $I_n = [a_{n+1}, b_n]$ is a reorganisation of the labelling of

$$\mathbb{R}, \{x_1, x_2, \ldots, x_n, \ldots | \forall n \in \mathbb{N}\}$$

it would seem that some $x_{n_0} \in \mathbb{R}$ in the diagram a_1 is **not in** I_n by the premise of the reorganisation one realises that an element x_n will be found in only one of the two disjoint closed intervals of I_{n+1},meaning when considering the intersection of all the intervals there will be no such $x_n \in \mathbb{R}$ which is found in every closed interval

$$\implies \bigcap_{n=1}^{\infty} I_n = \emptyset \ \text{※}$$

This notion **contradicts** The Nested Interval Property of the real numbers (**Theorem 1.1.1**) and shows that such a labelling and so **bijection onto** $\mathbb{N}$ is not possible. Thus $\mathbb{R}$ is shown to be **uncountable**. □

Cantor's Diagonal Argument
Theorem 1.2.7. *The open interval* $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ *is uncountable.*

Proof. The proof will proceed by contradiction.
Recalling the definition of countability, for contradiction, it will be assumed that

$$\exists\, f : \mathbb{N} \to (0,1),\ \forall x \in (0,1)\, \exists\, n \in \mathbb{N} : f(n) = x$$

Where the condition at the end of the statement is surjectivity. Extending this further using the intuition that $(0,1)$ it may also be assumed that a function of this sort is also injective and so bijective.
This means that we can enumerate or **label** each number in between 0 and 1. Consider the decimal representation of these numbers $a_{[m,n]}$ where m is the enumeration number and n is the decimal place of a number within that m-th decimal representation of the number in between 0 and 1. So for example $a_{5,7}$ would be the number in the 7th decimal place of the 5th enumerated real number in between 0 and 1. Having constructed such a system for organising and representing the numbers between 0 and 1 let's visualise them

$m \in \mathbb{N}$	$f(m)$	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	$\dots$
1	$f(1)$	$\mathbf{a_{1,1}}$	$a_{1,2}$	$a_{1,3}$	$\dots$
2	$f(2)$	$a_{2,1}$	$\mathbf{a_{2,2}}$	$a_{2,3}$	$\dots$
3	$f(3)$	$a_{3,1}$	$a_{3,2}$	$\mathbf{a_{3,3}}$	$\dots$
4	$f(4)$	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now we must show that there is some $a_{m,[n]}$ which can not be represented in this list. To do such we implement the following condition to define some number in between 0 and 1. Let this number be called b such that its decimal digit will be represented

$$b_i = \begin{cases} 1 & \text{if } a_{m,m} \neq 1 \\ 2 & \text{if } a_{m,m} = 1 \end{cases}$$

This means that **by construction** the first decimal place of b will be different from the first decimal place of $f(1)$ and the second decimal place of b will be different from the second decimal place of $f(2)$

$$\implies b_m \neq a_{m,m}$$

$$\implies b \neq f(m)\ \forall\, m \in \mathbb{N} ※$$

b is not in the list !
Thus there is no surjective function from $\mathbb{N}$ to the numbers between 0 and 1.
By contradiction. $\square$

Cantor's Theorem

Theorem 1.2.8. *Given any set A, there does not exist a function $f : A \to \mathcal{P}(A)$ which is surjective.*

What we're saying in terms of cardinality is that $|A| \ngeq \mathcal{P}(A)$ if a surjective function, which is perhaps obvious from the definition of the *power set* but let's show this rigorously.

Proof. The proof will proceed by contradiction.
Recalling the definition of countability, for contradiction, it will be assumed that

$$\exists\ f : A \to \mathcal{P}(A),\ \forall x \in \mathcal{P}(A)\, \exists\ a \in A : f(a) = x$$

Where the condition at the end of the statement is surjectivity.Now this means that as before we should be able to relate every $a \in A$ to some subset of A found in $\mathcal{P}(A)$ by the supposition.
For contradiction, we must show that there is some subset of A in $\mathcal{P}(A)$ which cannot be mapped to some $a \in A$ so that this subset would not be found in the image of the function.
Consider the subset

$$B = \{a \in A : a \notin f(a)\}$$

This subset is constructed of the elements in A which map to subset in $\mathcal{P}(A)$ which does not contain a itself. Thus **by construction** of B it is evident that B is not in the image of the function

$$\implies b \notin f(a)\ \forall\ b \in B \text{ ※}$$

□

Chapter 2

Sequences and Limits

2.1 Some Motivation

A well-known mathematician (while still a child) interpreted the meaning of the sum of an infinite geometric series in the following way.

There was a type of chocolate which was popularized by putting a coupon in the wrapping, and anyone who could produce 10 such coupons would get another bar of chocolate in exchange. If we have such a bar of chocolate, what is it really worth? Of course it is worth more than just one bar of chocolate since there is a coupon in it, and for each coupon you can get 1/10 of a bar of chocolate. But with this 1/10 of a bar will go one tenth of a coupon, and if for one coupon we get 1/10 of a bar of chocolate, for 1/10 of a coupon we get 1/100 of a bar of chocolate. To this 1/100 of a bar of chocolate belongs 1/100 of a coupon, and for this we again get 1/10 as much chocolate, i.e. 1/1000 of a bar of chocolate, and so on... This process can go infinitely, so that my one bar of chocolate together with its coupon is in fact worth 1+1/10+1/100+1/1000+... bars of chocolate. On the other hand, this is exactly 10/9 of a bar of chocolate. 1 is the value of the actual chocolate and the coupon that goes with it is 1/9 of a bar of chocolate. Indeed 9 coupons are worth one bar of chocolate because you could ask for such a bar and tell that you will pay after you eat it. So you eat it, take out the accompanying coupon, and give 10 coupons back which is exactly what is needed for one chocolate bar.

R. Péter, Playing with Infinity

If you like chocolate, then you need to like series too ! Not just any type of series either, infinite series, and to understand those we need to first understand finite series and that's where we'll start down here.

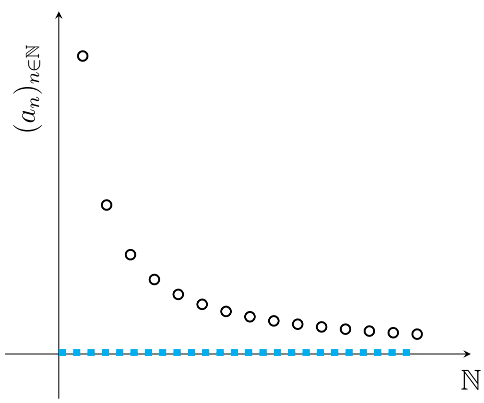
2.2 The Limit of a Series

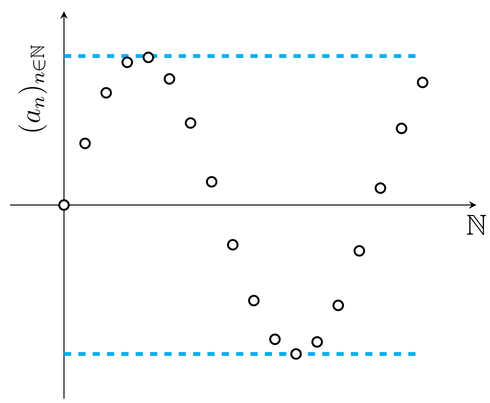
2.2.1 Formalisms

Definition 2.2.1. A *sequence* is a function whose domain is $\mathbb{N}$.

This definition means that a sequence is a list of labelled or **ordered** numbers echoing back to the many cases of labelling by $f : \mathbb{N} \to \mathbb{R}$ that were carried out in proofs from the previous section. In other words, a labelled function such that the nth term of this list is $f(n)$, can be called a sequence.
Similarly $f : \mathbb{N} \to \mathbb{C}$ would describe a complex sequence.

Notation To refer to some term in a sequence the notation $a_n = f(n)$ is used whilst the notations $(a_n)_{n\in\mathbb{N}}$ or $\langle a_n \rangle$ for the whole sequence.





The picture above on the left is a visualisation of the $\langle \frac{1}{n} \rangle$ sequence whilst on the right is a visualisation of a sine sequence. What is important to note is the evidence that the domain is $\mathbb{N}$ and thus discrete whilst the domain is part of $\mathbb{R}$

2.2.2 Convergence of a Sequence

Definition 2.2.2. A sequence $(a_n)_{n\in\mathbb{N}}$ converges to a real number a , referred to as the limit, if for every positive number ϵ, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

This phenomenon is denoted $\lim a_n = a$ or $a_n \to a$ for (a_n) which converges to the limit of a.
In light blue above, two cases of convergence are given.

On the left the convergence is clear such that $\lim_{n\to\infty} \frac{1}{n} = 0$ but for the case of the sine series, its *subsequences* converge to two distinct limits meaning that the function overall does not converge ! More on this in sections to follow

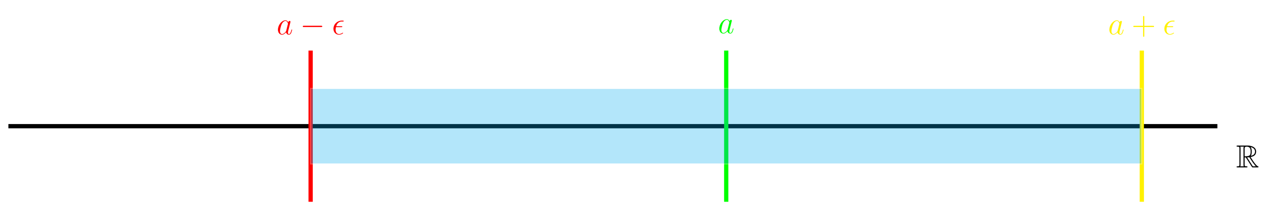
The definition of convergence as given above seems a bit difficult to digest on a first read. The best route to take towards understanding this concept is to understand what $|a_n - a| < \epsilon$ is.

Definition 2.2.3. - Neighbourhood

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the ϵ-neighbourhood of a



So an ϵ-neighbourhood if a contains all the points which are at most ϵ away from a or in other terms creates an interval with radius ϵ about a.

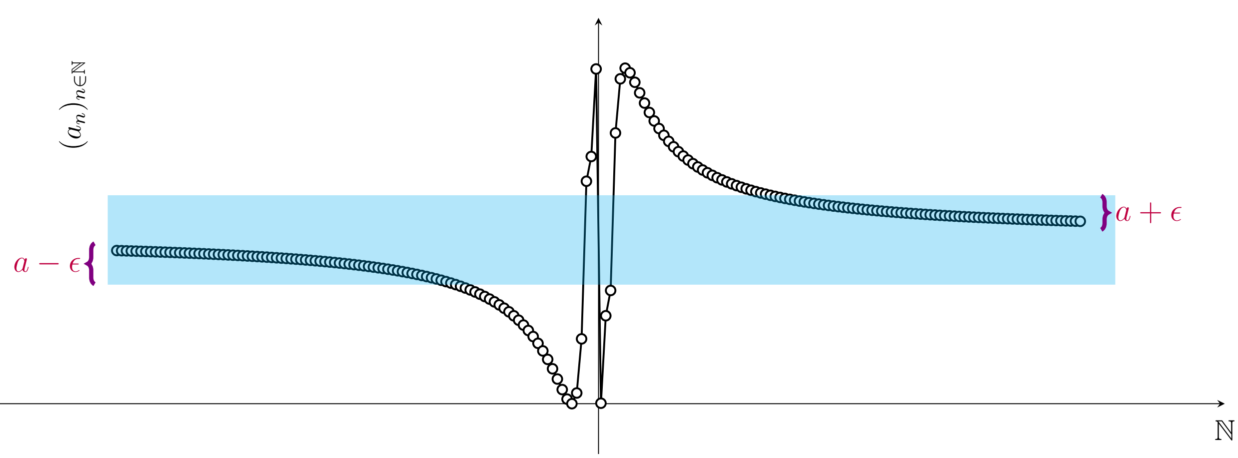
From this definition it is becoming clear that convergence is thus the **accumulation** of the points of a function about some point within its ϵ-neighbourhood. Let's state this more formally leading to a definition of convergence which is perhaps a bit more pictorial.

It should be noted that making use of **section 1.1.7** it is evident that in $\mathbb{C}$ this definition will be equivalent using a **Open Disc**.

Definition 2.2.4. A topological take on Convergence

A sequence (a_n) converges to a if, given any ϵ-neighbourhood $V_\epsilon(a)$, there exists a point in the sequence,a_N, after which all proceeding terms are found within $V_\epsilon(a)$.

As a resuslt, ever ϵ-neighbourhood of a converging sequence contains all but a finite number of terms of (a_n).



Negating the Definition of Convergence As given above the definition of convergence tells us that some sequence (a_n) is convergent to some limit a if for all points greater than some point N in its domain, the sequence occupies an ϵ-neighbourhood around a.

$$(a_n) \to a \iff \forall\, n \geq N : N \in \mathbb{N}, |a_n - a| < \epsilon.$$

What's the negation of this statement ? What does it mean for a sequence to not converge to some point. This is what the negation of this statement tell us. Mathematically this is straightforward

$$(a_n) \not\to a \iff \forall\, N \in \mathbb{N},\ \exists\, n \geq N : |a - a_n| \geq \epsilon.$$

But what this means conceptually is that we will always be able to find at least one point which exists outside an ϵ-neighbourhood for any choice of N.

2.2.3 Null Sequences

Definition 2.2.5. If $\lim_{n\to\infty} a_n = 0$, the sequence (a_n) is called a null sequence meaning

$$\forall\, \epsilon > 0, \exists N \in \mathbb{N} : |a_n| = |a_n - 0| < \epsilon\ \forall\, n > N$$

Example 1 Consider the sequence (a_n), where $a_n = \frac{1}{n}$, let's show this is a null sequence.

Proof. Let $\epsilon < 0$ be arbitrary.
To proceed we recall what convergence means and the notion of an epsilon neighbourhood. By the **Archimidean Property** of $\mathbb{R}$ N is selected such that there will always be a number N such that $\frac{1}{N}$ is smaller than ϵ.

$$\implies \frac{1}{N} < \epsilon$$

Now consider that for convergence n must be within the epsilon neighbourhood where points are smaller or equal to N

$$\begin{aligned} \implies \frac{1}{n} \leq \frac{1}{N} < \epsilon \\ \frac{1}{n} < \epsilon \\ \frac{1}{n} - 0 < \epsilon \\ \implies |a_n - 0| < \epsilon \\ \therefore \lim_{n\to\infty} a_n = a = 0 \end{aligned}$$

□

The Approach When proving that a function converges to a given limit we always start by saying "Let $\epsilon > 0$ be arbitrary". After this comes the most crucial but perhaps initially counter intuitive step as we start by considering where we want to end up for the proof.

$$|a_n - a| < \epsilon$$

is considered and using this N is chosen. Having now found this N we legitimise our choice using n.
The inequality is then derived.

Let's try this method out in some more examples.

Example 2 Consider the sequence (a_n), where $a_n = \frac{1}{\sqrt{n}}$, let's show this is a null sequence.

What we want to end up with from the definition of convergence is clearly

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \epsilon.$$

This infers that a good starting point for the choice of N would be

$$\frac{1}{\sqrt{N}} < \epsilon$$
$$\frac{1}{N} < \epsilon^2$$
$$\frac{1}{\epsilon^2} < N.$$

With this in hand we can start the proof.

Proof. Let $\epsilon > 0$ be arbitrary.
Choose $N \in \mathbb{N}$ satisfying

$$\frac{1}{\epsilon^2} < N$$

as the boundary of the epsilon neighbourhood of convergence. Affirming this, let $n \geq N$

$$\frac{1}{\epsilon^2} < n$$
$$\frac{1}{\sqrt{n}} - 0 < \epsilon$$
$$\implies \left|\frac{1}{\sqrt{n}} - 0\right| < \epsilon$$
$$\therefore \lim_{n\to\infty} a_n = a = 0$$

□

Example 3 Show that the sequence $a_n = \lim \frac{2}{\sqrt{n+3}} = 0$

As is typical we start from what we know, which is what we want to attain a choice of N.
Using the definition of convergence and the knowledge that the sequence is allegedly null

$$\left|\frac{2}{\sqrt{n+3}}\right| < \epsilon.$$

This would mean that N the boundary of the ϵ-neighbourhood would be of the form

$$\left|\frac{2}{\sqrt{N+3}}\right| < \epsilon$$
$$\left|\frac{4}{N+3}\right| < \epsilon^2$$
$$\frac{4}{\epsilon^2} - 3 < N.$$

With this choice in mind we now move on to the proof.

Proof. Let $\epsilon > 0$, arbitrarily.
Choose $N \in \mathbb{N}$ to be $\frac{4}{\epsilon^2} - 3 < N$.
For affirmation let $n \geq N$

$$\implies \frac{4}{\epsilon^2} - 3 < n$$
$$\frac{4}{\epsilon^2} < n + 3$$
$$\frac{4}{n+3} < \epsilon^2$$
$$|\frac{2}{\sqrt{n+3}} - 0 < \epsilon|$$
$$\therefore \lim \frac{2}{\sqrt{n+3}} = 0$$

□

Example 4 - Applying $\mathbf{\frac{1}{n}}$ Show that the sequence $a_n = \lim \frac{n+1}{n} = 1$

We want to end up with

$$\left|\frac{n+1}{n} - 1\right| < \epsilon$$

Reformulating the term on the left

$$\left|\frac{n+1-n}{n}\right| = \left|\frac{1}{n}\right| < \epsilon$$

$$\frac{1}{\epsilon} < n$$

Proof. Let $\epsilon > 0$, arbitrarily.
Choose $N \in \mathbb{N}$ to be $\frac{1}{\epsilon} < N$.
For affirmation let $n \geq N$

$$\begin{aligned} \implies \frac{1}{\epsilon} &< n \\ \frac{1}{n} &< \epsilon \\ \left|\frac{1}{n} - 0\right| &< \epsilon \end{aligned}$$

Thus it is evident that $\left|\frac{n+1}{n} - 1 - 0\right| < \epsilon$ and so the limit is indeed 1 given that when 1 is taken to LHS it became 0. □

The size of the ϵ-neighbourhood of non-null sequences and convergence within half of the limit
Theorem 2.2.1. *If* $\lim_{n\to\infty} a_n = a \neq 0$, *then* $\exists\, N \in \mathbb{N}$:

$$|a_n| > \frac{|a|}{2}\, \forall\, n \geq N.$$

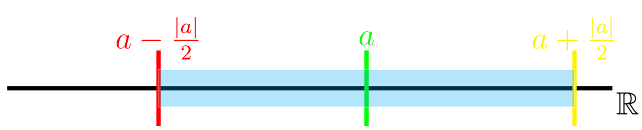
Moreover, if $a > 0$, *then one has*

$$a_n > \frac{a}{2}\, \forall\, n \geq N,$$

whereas if $a < 0$, *then one has*

$$a_n < \frac{a}{2}\, \forall\, n \geq N.$$

At first glance the theorem seems somewhat abstruse but let's break it down into bits and pieces understanding it in parts. The first bit of information that is derived from the condition is that the theorem applies to sequences which are not null and thus have some limit, $a \neq 0$. The theorem then posits that as a result of the sequence not being null the size of some edge point, a_N, of the ϵ-neighbourhood must be smaller than half the size of the limit.
Thinking of this from another perspective one can interpret this as having the distance between a and a_N be at the least $\frac{|a|}{2}$



This is indicative of the ϵ value which will lead to the proof.
The points which follow are cases of the preceding statement which should be fairly easy to proof. Now that we understand what's going on, we're ready to begin !

Proof.

Using the reasoning presented above, let $\epsilon = \frac{|a|}{2}$ since $a \neq 0$, by definition of convergence $\exists\ N \in \mathbb{N} : \forall\ n \geq N$

$$|a_n - a| < \frac{|a|}{2}. \tag{2.1}$$

Now consider

$$|a| = |a - a_n + a_n| \tag{2.2}$$

by applying the triangle equality

$$|a - a_n + a_n| \leq |a - a_n| + |a_n|. \tag{2.3}$$
$$\implies |a| \leq |a - a_n| + |a_n| \tag{2.4}$$
$$\implies |a| - |a_n| \leq |a - a_n| \tag{2.5}$$

Making use of the ϵ relation from (2.1)

$$\implies |a| - |a_n| \leq |a_n - a| < \frac{|a|}{2} \tag{2.6}$$

Initially line (2.6) may seem incorrect but we're just talking about distance so whilst ex: $a_n - a$ may give a positive value and $a - a_n$ may give a negative value their modulus and hence what we talk about when we talk about distance is the same! And so, this type of value swapping is permissible.

$$\implies -|a_n| < \frac{|a|}{2} - |a| \tag{2.7}$$
$$|a_n| > |a| - \frac{|a|}{2} \tag{2.8}$$
$$\therefore |a_n| > \frac{|a|}{2} \tag{2.9}$$

That's the first part done now let's prove the cases! Before we can do this a relation must be derived where a_n is considered and not $|a_n$.
Starting with the ϵ relation

$$|a_n - a| < \frac{|a|}{2}.$$

By definition of ϵ-neighbourhood this may be reformulated as

$$-\frac{|a|}{2} < a_n - a < \frac{|a|}{2} \implies a - \frac{|a|}{2} < a_n < \frac{|a|}{2} + a$$

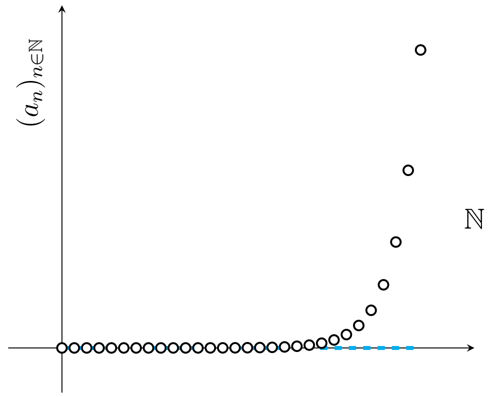
Thus,

$$\begin{cases} a > 0 \implies a_n > \frac{a}{2}\, \forall\, n \geq N \\ a < 0 \implies a_n < \frac{a}{2}\, \forall\, n \geq N \end{cases}$$

□

2.2.4 Informal Divergence

Definition 2.2.6. A sequence that does not converge is said to **diverge**.



This notion completely opposes that of converges such that points within a converging series get farther away from each other after some N.
This gives us a vague idea of what divergence is but let's investigate this notion by considering the series

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, ...\right).$$

The definition that has been ascribed to convergence is intrinsically reliant on there being an $N \in \mathbb{N}$ for every $\epsilon > 0$. Now given that divergence is defined as the negation of convergence then within a divergent series there does not exists an $N \in \mathbb{N}$ for every $\epsilon > 0$.
From a graphical perspective this would mean that there are ϵ-neighbourhoods within a series which cannot be evidently specified by some N.

The series given above in parentheses clearly has ϵ-neighbourhoods which have a corresponding N. $\epsilon = \frac{1}{2}$ corresponds to $N = 3$ but what if we consider $\epsilon = \frac{1}{10}$?
If this were the case the series would converge at some value of either $\frac{1}{5}$ or $\frac{1}{5}$, leading to two cases

$$\left|\frac{1}{5} - a\right| < \frac{1}{10} \qquad\qquad \left|-\frac{1}{5} - a\right| < \frac{1}{10}$$

By applying the triangle inequality and subtracting the two situations together

$$0 = \left|\frac{1}{5} + \frac{1}{5}\right| = \left|\frac{1}{5} - a + a + \frac{1}{5}\right| \leq \left|\frac{1}{5} - a\right| + \left|\frac{1}{5} + a\right| < \frac{1}{10} + \frac{1}{10} = \frac{1}{5} ※$$

Thus, it is clear that for $\epsilon = \frac{1}{n}$, N is unknown and such a sequence is divergent.

Further Example Show that the sequence

$$(-3, -2, -1, 0, 1, 0, 1, 0, 1, ...)$$

is divergent.

Proof. Proceeding by applying the approach from the expository problem we assume that the sequence is convergent to some ϵ-neighbourhood for some $N \in \mathbb{N}$ and then show that this is a contradiction and that so not every $\epsilon < 0$ has an associated $N \in \mathbb{N}$. Thus giving evidence for divergence.

Assuming convergence, consider $\epsilon = \frac{1}{3}$ which by the assumption means that

$$\exists\ N \in \mathbb{N} : \text{ for } n \geq N, |a_n - a| < \frac{1}{3}$$

Now it is evident that for any N the series will take up values of 0 or 1

$$|1 - a| < \frac{1}{3} \qquad\qquad |0 - a| < \frac{1}{3}$$

By applying the triangle equality and subtracting the two

$$1 = |1 - 0| = |1 - a + a - 0| \leq |1 - a| + |a - 0| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \text{ ※}$$

□

2.3 Boundedness and Limit Theorems

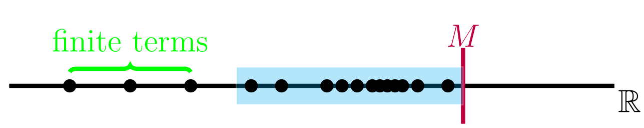
The study of convergent sequences is motivated by a want to understand the behaviour of a limit.
One way we can do this is by examining the boundedness of a sequence.
Much of what is covered in this section is informed by what was discussed in sections 1.1.3 & 1.1.4.

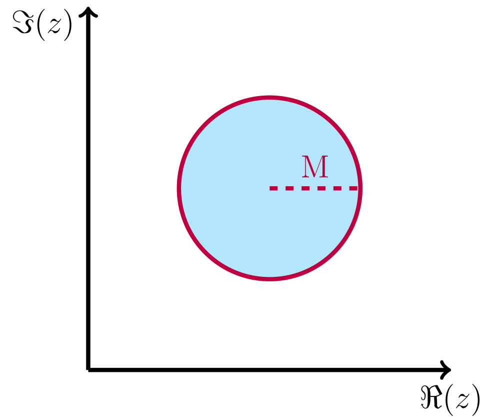
2.3.1 Convergent Series are Bounded

Definition 2.3.1. A sequence (a_n) is said to be *bounded* if there exists a number $M > 0$ such that $|a_n| \leq M\ \forall\ n \in \mathbb{N}$
Theorem 2.3.1. *Every convergent sequence is bounded*

First let's think about this about what it means to be convergent. Convergent sequences are those which after some point in the sequence a_N will become confined to some ϵ-neighbourhood.
This **confinement** is within its very nature, as depicted in the figures above with ϵ-neighbourhoods in blue, indicative of some maximum point.

Proof. The proof will follow with this notion in mind using the definition of convergence.





Consider the sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = a$ and taking $\epsilon = 1$, by definition of convergence $\exists\ N \in \mathbb{N} : \forall\ n \geq N$

$$|a_n - a| < 1.$$

With this relation in hand, consider now

$$|a_n| = |a_n - a + a|$$

which by applying the traingle inequality

$$|a_n - a + a| \leq |a_n - a| + |a|.$$

By the ϵ-neighbourhood relation derived in the preceding lines

$$|a_n - a| + |a| < |a| + 1$$

And so

$$|a_n| < |a| + 1.$$

This shows that the points in the ϵ-neighbourhood are strictly bounded by $|a|+1$ but the argument made exclude the finite terms which do not coincide in the ϵ-neighbourhood such that they are labelled by

$$n < N.$$

Accounting for these terms;

$$\text{let } M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 1\}.$$

$$\therefore |a_n| \geq M\ \forall\ n \in \mathbb{N}$$

□

It is good to note that reversing this argument is not consistent.
Boundedness is a necessary condition for convergence to occur but it is not within itself a sufficient condition for a sequence to converge.

2.3.2 Algebraic Limit Theorem

Now that an understanding of what limits are has been achieved one need to start understanding some basic rules of operations related to and making use of limits. This is what The Algebraic Limit Theorem is about.

Theorem 2.3.2. *Let* $\lim a_n = a$, *and* $\lim b_n = b$. *Then,*

1. $\lim(ca_n) = ca \, \forall \, c \in \mathbb{R}$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n b_n) = ab$
4. $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ *given* $b \neq 0$

Proof. For *1*.
Starting from what we know to attain a choice of N.
By definition of convergence it is evident that

$$|ca_n - ca| < \epsilon \; \exists \; \epsilon > 0. \tag{2.10}$$

This can be rearranged

$$|c||a_n - a| < \epsilon \tag{2.11}$$

$$\implies |a_n - a| < \frac{\epsilon}{|c|} \tag{2.12}$$

This relation being achieved we want to choose an N such that $\forall \, n \geq N$

$$|a_n - a| < \frac{\epsilon}{|c|} \text{ occurs.} \tag{2.13}$$

In affirmation consider the initial form

$$|ca_n - ca| = |c||a_n - a| \tag{2.14}$$

Now implementing (2.13) let $n \geq N \, \forall \, n$

$$|c||a_n - a| < |c|\frac{\epsilon}{|c|} \tag{2.15}$$

$$\therefore \; |c||a_n - a| < \epsilon \tag{2.16}$$

The case has thus shown to hold $\forall \, c \in \mathbb{R}$ except for 0 where it reduces to a sequence of 0s which converges to 0 showing that 1 holds indeed for all cases.

For *2*.
A similar preliminary exercise to suss out the choice of N must be carried out.
By definition of convergence we wish to end up with

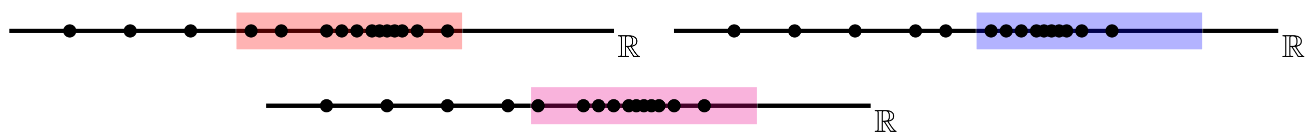
$$|a_n + b_n - (a + b)| < \epsilon \; \exists \; \epsilon > 0$$

$$|a_n - a + b_n - b)| < \epsilon.$$

Now applying the triangle inequality this interval can be split

$$|a_n - a| + |b_n - b)| < \epsilon.$$

To move forward, by the hypothesis that $a_n \to a$ and $b_n \to b$, we split the ϵ such



that two points exist corresponding to convergence N_1 and N_2 relating to each limit, respectively. Giving;

$$|a_n - a| < \frac{\epsilon}{2} \text{ when } n \geq N_1 \qquad\qquad |b_n - b| < \frac{\epsilon}{2} \text{ when } n \geq N_2$$

For $n \geq \{N_1, N_2\}$ we choose $N = \max\{N_1, N_2\}$ such that $\forall\, n \geq N$

$$|a_n - a| < \frac{\epsilon}{2} \qquad\qquad |b_n - b| < \frac{\epsilon}{2}.$$

In affirmation consider the initial form

$$|a_n + b_n - (a + b)|.$$

As earlier by the triangle equality

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b|$$

and now by the hypothesis of $n \geq \{N_1, N_2\}$

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus the sum of two neighbourhoods is the sum of their limits.

For *3*.
Again we start by finding a way to choose N. We wish to end up with

$$|a_n b_n - ab| < \epsilon \, \exists \, \epsilon > 0.$$

Once more we wish to separate the interval and apply the triangle inequality. Consider the following to move in that direction

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| < \epsilon \text{ Midpoint Trick}$$

$$|a_n b_n - ab_n + ab_n - ab| = |b_n(a_n - a) + a(b_n - b)| < \epsilon.$$

Thus by the triangle inequality

$$|b_n(a_n - a) + a(b_n - b)| \leq |b_n||a_n - a| + |a||b_n - b|$$

As within *2.* we wish to once more split the inequality making each part less than $\frac{\epsilon}{2}$. For $|a||b_n - b|$ this will be simple as before given that $|a|$ is constant. As such there exists some N_1 such that when $n \geq N_1$

$$|a||b_n - b| < \frac{\epsilon}{2}$$

$$|b_n - b| < \frac{\epsilon}{2|a|}.$$

Now, for $|b_n|a_n - a|$ the case is now more complex as b_n is a **variable** and not a constant. What we wish to do is make $|b_n|$ less than some positive $\frac{\epsilon}{2}$ and recalling **Theorem 2.3.1** with the hypothesis that $b_n \to b$ then b_n must have some upper bound, say $M : b_n \leq M \, \forall \, n \in \mathbb{N}$. With this in hand we would be indeed *safe* choosing some N_2 such that whenever $n \geq N_2$

$$|a_n - a| < \frac{\epsilon}{2M}$$

For $n \geq \{N_1, N_2\}$ we choose $N = \max\{N_1, N_2\}$ such that $\forall \, n \geq N$

$$|a_n - a| < \frac{\epsilon}{2M} \qquad\qquad |b_n - b| < \frac{\epsilon}{2|a|}$$

In affirmation consider the initial form

$$|a_n b_n - ab|$$

And applying the Midpoint Trick

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| = |b_n(a_n - a) + a(b_n - b)|.$$

By the triangle inequality

$$|b_n(a_n - a) + a(b_n - b)| \leq |b_n||a_n - a| + |a||b_n - b|.$$

Now implementing the upper bound of b_n

$$|b_n||a_n - a| + |a||b_n - b| \leq M|a_n - a| + |a||b_n - b|$$

By the hypothesis of the choice of N, let $n \geq N \, \forall n \in \mathbb{N}$

$$\implies M|a_n - a| + |a||b_n - b| < \frac{M\epsilon}{2M} + \frac{|a|\epsilon}{2|a|}$$

$$\implies M|a_n - a| + |a||b_n - b| < \epsilon$$

$$\therefore |a_n b_n - ab| < \epsilon \text{ as required.}$$

For *4*.
This proof is based 3. obviously, with the difference that we will have to prove that

$$\frac{1}{b_n} \to \frac{1}{b}.$$

Starting out, as always we start by considering where we want to end up which for this case will be

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \epsilon \, \exists \, \epsilon > 0. \tag{2.17}$$

Applying a midpoint trick as in *3*. we manipulate the interval

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n}{b_n} - \frac{a}{b_n} + \frac{a}{b_n} - \frac{a}{b}\right| = \left|\frac{1}{b_n}(a_n - a) + a\left(\frac{1}{b_n} - \frac{1}{b}\right)\right|. \tag{2.18}$$

The triangle inequality may now be applied

$$\left|\frac{1}{b_n}(a_n - a) + a\left(\frac{1}{b_n} - \frac{1}{b}\right)\right| \leq \left|\frac{1}{b_n}\right| |a_n - a| + |a| \left|\frac{1}{b_n} - \frac{1}{b}\right| < \epsilon. \tag{2.19}$$

As before we split ϵ

$$\left|\frac{1}{b_n}\right| |a_n - a| < \frac{\epsilon}{2} \qquad\qquad |a| \left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{\epsilon}{2}. \tag{2.20}$$

Earlier when we had $|b_n|$ we considered some upper bound to deal with the worst case scenario of the variable but now for $|\frac{1}{b_n}|$ we must consider a lower bound such that the terms in the sequence are close to b.

Recall **Theorem 2.2.1** which showed that convergence occured within a distance of half the limit, from the limit. Applying this theorem here under the assumption that $b_n \to b$

$$\implies |b_n| > \frac{|b|}{2} \tag{2.21}$$

$$\left|\frac{1}{b_n}\right| < \frac{2}{|b|}. \tag{2.22}$$

With this inequality in hand we can choose N_1 for a_n such that $\forall \, n \geq N_1$

$$\left|\frac{1}{b_n}\right| |a_n - a| < \left|\frac{2}{b}\right| |a_n - a| < \frac{\epsilon}{2} \tag{2.23}$$

$$\implies |a_n - a| < \frac{|b|\epsilon}{4} \tag{2.24}$$

Similarly this can be applied to b_n for a the choice of N_2 such that $\forall\, n \geq N_2$

$$|a|\left|\frac{1}{b_n} - \frac{1}{b}\right| = |a|\left|\frac{b - b_n}{b_n b}\right| = |a|\left|\frac{b_n - b}{b_n b}\right| < \frac{\epsilon}{2} \tag{2.25}$$

Applying the relation from line (2.21) once more;

$$\implies |a|\left|\frac{b_n - b}{b_n b}\right| < 2|a|\left|\frac{b_n - b}{b^2}\right| < \frac{\epsilon}{2} \tag{2.26}$$

$$\implies |b_n - b| < \frac{\epsilon|b|^2}{4|a|} \tag{2.27}$$

Thus we choose $N = \max N_1, N_2$, in affirmation consider line (2.19) and let $n \geq N \,\forall\, n$

$$\left|\frac{1}{b_n}\right||a_n - a| + |a|\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{1}{b_n}\right||a_n - a| + |a|\left|\frac{b_n - b}{b_n b}\right| \tag{2.28}$$

Once more applying **Theorem 2.2.1**

$$\left|\frac{1}{b_n}\right||a_n - a| + |a|\left|\frac{b_n - b}{b_n b}\right| < \left|\frac{2}{b}\right||a_n - a| + 2|a|\left|\frac{b_n - b}{b^2}\right| \tag{2.29}$$

Now making use of the choice of N for $n \geq N$

$$\left|\frac{2}{b}\right||a_n - a| + 2|a|\left|\frac{b_n - b}{b^2}\right| < \left|\frac{2}{b}\right|\frac{|b|\epsilon}{4} + 2|a|\left|\frac{b^2\epsilon}{4ab^2}\right| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{2.30}$$

□

Uniqueness of a Limit

Lemma 2.3.3. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$*

Let's take a step back and understand what this lemma is saying. It proposes that if two points are equal then this is the same as saying that the distance between said to points will always be less than some positive number, epsilon, for all epsilon.

> If equal then one can always find a positive number which the distance between the two numbers is less than.

Proof. Given that this statement is biconditional it will be proven in two steps.
For ($\implies$)
This case is trivial as $|a - a|$ evidently gives $|0|$ which is less than all ϵ for $\epsilon < 0$.

For ($\Longleftarrow$)
Now for the interesting part. We take an approach by contradiction and assume that $a \neq b$ and consider the statement

$$|a-b| < \epsilon \,\forall\, \epsilon > 0.$$

But if $a \neq b$ from the assumption then

$$\exists\ \epsilon_0 = |a-b| > 0 \text{ ※}.$$

Thus $a = b$. □

Theorem 2.3.4. *If a sequence converges, then the limit is unique.*

Proof. Let $\lim a_n = l_1$ and also $\lim a_n = l_2$.
By definition of convergence this means that there exists two distinct points, say N_1 and N_2 such that when n is greater than either one the sequence is confined to some neighbourhood.

$$\forall n \geq N_1, |a_n - l_1| < \frac{\epsilon}{2}, \forall \epsilon > 0 \qquad \forall n \geq N_2, |a_n - l_2| < \frac{\epsilon}{2}, \forall \epsilon > 0$$

By intuition of **Lemma 2.3.3.** we choose $N = \max\{N_1, N_2\}$ such that $\forall\, n \geq N$

$$|l_1 - l_2| = |l_1 - a_n + a_n l_2|$$

By application of a midpoint trick. Now, making use of the triangle inequality

$$|l_1 - a_n + a_n - l_2| \leq |a_n - l_1| + |a_n - l_2|$$

Which by the choice of N gives

$$|a_n - l_1| + |a_n - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\implies |l_1 - l_2| < \epsilon$$

Which by **Lemma 2.3.3.** shows that l_1 and l_2 are equal. □

Alternatively the proof can follow by an application of the Algebraic Limit Theorem.

Proof.

$$\lim(a_n - a_n) = \lim(a_n) - \lim(a_n) = l_1 - l_2$$

But $\lim(a_n - a_n) = 0$

$$\implies l_1 - l_2 = 0$$

$$\therefore l_1 = l_2$$

□

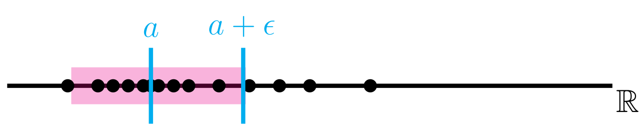
2.3.3 Order Limit Theorem

In the previous section we examined the algebraic manipulation of sequences, showing that algebraic manipulation still holds throughout the process of taking a limit.
In this section it will be shown that the process of taking limits of sequences holds throughout for order relations.

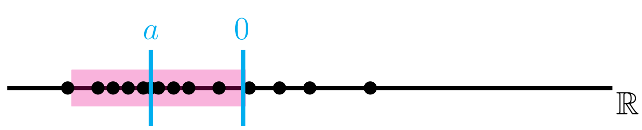
Theorem 2.3.5. *Let* $\lim a_n = a$*, and* $\lim b_n = b$*. Then,*

1. *If* $a_n \geq 0\, \forall n \in \mathbb{N}$ *then* $a \geq 0$
2. *If* $a_n \leq b_n\, \forall n \in \mathbb{N}$ *then* $a \leq b$
3. *If* $\exists\ c \in \mathbb{R}$ *for which* $c \leq b_n\, \forall\, n \in \mathbb{N}$ *then* $c \leq b$*.*
   Similarly, if $a_n \leq c\, \forall\, n \in \mathbb{N}$*, then* $a \leq c$*.*

Proof. For *1*.
What we're saying here is that if a convergent sequence is positive, then it should converge to a positive limit also. Pretty straight forward right.
Taking an approach by contradiction, assume that $a_n > 0$ has a negative limit $a < 0$.



Considering the fact that $a < 0$ this would mean that taking $\epsilon = |a|$ would give $a+\epsilon = 0$.



This implies that by definition of convergence there exists some N such for all $n \geq N$

$$|a_n - a| < 0.$$

But particularly this infers that

$$|a_N - a| < 0.$$

$$\text{Showing that } a_N < 0 \text{ ※}.$$

This means that not all terms in the sequence are positive leading to a contradiction and thus proving that a positive sequence must have a positive limit.

For *2*. From the Algebraic limit theorem it is understood that the sequence $(b_n - a_n)_{n\in\mathbb{N}}$ converges to $b_n - a_n$.
Given that $b_n > a_n$ this implies that $b_n - a_n > 0$ which means that $b - a > 0$ from *1*.
Thus $b > a$ as required.

For *3*. This follows directly from *2* where a_n is a sequence in this case.

$$a_n \leq b_n \text{ then } a \leq b$$

Taking $a_n = c$ a constant and not a sequence;

$$c \leq b_n \implies c \leq b$$

□

Tails In Analysis it tends to be the case that we are only interested in what happens towards the of a function and not at what happens for the first, say 10,000 n. This creates an issue of weakness in statement regarding order in the sense that we show that the statement hold for sequences that *eventually* become non-negative. The theorem stated still remain valid but may be tweaked to include some N_1 after which all elements of a sequence will be positive for example, making the theorem more accurate.

Sandwich Lemma or Squeeze Theorem
Theorem 2.3.6. *If* $x_n \leq y_n \leq z_n$ *and also* $\lim x_n = l = \lim z_n = l$ *then* $\lim y_n = l$ *also.*

Proof. Let $\epsilon > 0$ be arbitrary.
By definition of convergence there exists some N_1 and N_2 such that

$$\forall\, n \geq N_1,\ |x_n - l| < \epsilon \qquad\qquad \forall\, n \geq N_2,\ |z_n - l| < \epsilon$$

Given that $x_n \leq y_n \leq z_n$, this suggests that choosing $N = \max N_1, N_2$ and taking all $n \geq N$ would give

$$|y_n - l| < \epsilon$$

This infers that x_n, y_n and z_n all occupy the same ϵ-neighbourhoods and converge to the same limit, showing they are equal.
As required. □

An Example *Calculate* $\lim_{n\to\infty} \frac{\sin n}{n^2}$.

My first inclination seeing a fraction, was to make use of the algebraic limit theorem, but consider that $\sin n$ is an oscillating sequence and thus has a problematic limit. As such we consider what we know about this case primarily that sine exists between

$$-1 \leq \sin n \leq 1.$$

So takes care of the numerator, now to get the denominator in we multiply throughout by $\frac{1}{n^2}$ giving

$$-\frac{1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}.$$

This is surely reminiscent of **Theorem 2.3.6.**, the sandwich lemma. As such we must consider the limit

$$\lim_{n\to\infty} \frac{1}{n^2}.$$

Now by the algebraic limit theorem, the following manipulation is permissible

$$\lim_{n\to\infty} \frac{1}{n^2} = \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right).$$

It's good to recall that $\lim_{n\to\infty} \frac{1}{n}$ was the first proof of convergence presented ! But let's go over it again for the sake of revision.
The sequence $\frac{1}{n}$ is composed of elements in $\mathbb{R}$ and by the Archimedean property of $\mathbb{R}$ there always exists some $N \in \mathbb{N}$ whose reciprocal will be smaller than some arbitrarily chosen, positive ϵ.

$$\frac{1}{N} < \epsilon$$

Now let $n \geq N$;

$$\implies \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

And so

$$\frac{1}{n} < \epsilon$$

which by definition of convergence defines an ϵ-neighbourhood

$$\left|\frac{1}{n} - 0\right| < \epsilon.$$

Thus $\frac{1}{n}$ is a null sequence giving

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

Applying the squeeze theorem we arrive to the result that

$$\lim_{n\to\infty} \frac{\sin n}{n^2} = 0.$$

An Example involving Sequence Shuffling Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Proof. This being a bidirectional statement the proof will proceed that both cases of the argument hold.

For ($\implies$).
Assume $(z_n) \to L$, $\exists\ L \in \mathbb{R}$, so by the definition of convergence for some arbitrary $\epsilon < 0$;

$$|z_n - L| < \epsilon.$$

$$\longrightarrow$$

$\underline{x_1}\,\underline{y_2}\,\underline{x_2}\,\underline{y_1}\ldots\ldots$

$\underline{x_1}\quad\underline{x_2}\quad\ldots\ldots$

$\quad\underline{y_2}\quad\underline{y_1}\ldots\ldots$

Now convergence occurs $\forall n \geq N, \exists N \in \mathbb{N}$ for (z_n) but consider that the point z_N may be described as $y_{\frac{n}{2}}$ in terms of (y_n) as it contains half as many points. Similarly, z_N may be described by x_n as the point preceeding $y_{\frac{n}{2}}$.

$$\implies z_N = y_{\frac{n}{2}} \qquad\qquad \implies z_N = y_{\frac{n}{2}} - 1 = x_n$$

As a result it must surely follow that in (y_n) there exists some point $N_1 \geq \frac{N}{2}$ such that for all $n \geq N_1$

$$|y_n - L| < \epsilon.$$

And for (x_n) there exists some point N_2 defining an ϵ-neighbourhood containing a point more than that of y_n such that $N_2 \geq \frac{N+1}{2}$. For all $n \geq N_2$

$$|x_n - L| < \epsilon \text{ as required for the forward implication.}$$

For ($\Longleftarrow$).
Assuming that $(x_n) \to L$ and $(y_n) \to L$ and letting an arbitrary $\epsilon > 0$. We have, by the definition of convergence, that there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ in (x_n)

$$|x_n - L| < \epsilon.$$

Similarly there exists some $N_2 in \mathbb{N}$ such that for all $n \geq N_2$ in (y_n)

$$|y_n - L| < \epsilon.$$

Now given that the sequence (z_n) is composed of (x_n) and (y_n) it stands to reason by construction that there exists some $N \in \mathbb{N}$ of the form $N = \max\{2N_1, 2N_2\}$, given that (z_n) contains twice as many points, which for all $n \geq N$ in (z_n)

$$|z_n - L| < \epsilon.$$

$\square$

An Example involving Sequences and Absolute Sequences

(a) Show that if $(b_n) \to b$, then the sequence of absolute values $|b_n|$ converges to $|b|$.

(b) Is the converse of part (a) true? If we know that $|b_n| \to |b|$, can we deduce that $(bn) \to b$?

Starting with (a) we are to carry out a proof of convergence. The tools at our disposal for absolute values is the definition and the triangle inequality. Certainly the latter is more malleable and will hopefully prove a good starting point.

Proof. Initially we consider were we want to end up;

$$||b_n| - |b|| < \epsilon \,:\, \epsilon < 0.$$

Starting with $|b_n|$ a midpoint trick may be applied giving

$$|b_n| = |b_n - b + b|$$

exposing the applicability of the triangle inequality in this case.

$$|b_n - b + b| \leq |b_n - b| + |b|$$

$$\implies |b_n| \leq |b_n - b| + |b|$$

Which gives

$$|b_n| - |b| \leq |b_n - b|.$$

Now, for $||b_n| - |b||$ by definition of absolute value we must also show that $|b| - |b_n| \leq |b_n - b|$. Similarly,

$$|b| - |b_n| \leq |b - b_n|$$

but given that this is a distance this inequality is equal to

$$|b| - |b_n| \leq |b_n - b|$$

thus giving

$$||b_n| - |b|| \leq |b_n - b|$$

By the initial premise that $|b_n - b| < \epsilon, \forall n \geq N \,:\, \exists\, N \in \mathbb{N}$ by definition of convergence we have;

$$||b_n| - |b|| \leq |b_n - b| < \epsilon$$

$$||b_n| - |b|| < \epsilon$$

□

For (b) consider that the absolute value is defined by a piecewise function and so does not capture the ordered nature of b_n. As a counterexample consider $b_n = (-1)^n$ which is divergent but $|b_n|$ is convergent to 1.

Where the Algebraic Limit Theorem Fails

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove the Algebraic Limit Theorem, part (iii), for the case when b = 0.

For (a) we present a proof of convergence.

Proof. We begin by finding a way to choose N.
By definition of convergence we wish to end up with

$$|a_n b_n - 0| < \epsilon : \exists\ \epsilon > 0.$$

Consider that since $|a \cdot b| = |a||b|$ we can rewrite this inequality as

$$|a_n||b_n| < \epsilon.$$

Now under the condition that (a_n) is bounded

$$\exists\ M\ :\ M \geq a_n \forall a_n \in (a_n : n \in \mathbb{N})$$

$$\implies |a_n||b_n| \leq M|b_n| < \epsilon$$

Thus this gives

$$|b_n| < \frac{\epsilon}{M}.$$

With this in hand we choose $N \in \mathbb{N}$ such that for all $n \geq N$ in $(a_n b_n)$

$$|b_n| < \frac{\epsilon}{M}.$$

In affirmation consider the initial form

$$|a_n b_n - 0| = |a_n||b_n|$$

Now considering that (a_n) is bounded by M and letting $n \geq N\,\forall\,n$

$$|a_n||b_n| < M\frac{\epsilon}{M} = \epsilon$$

$$\therefore\ |a_n b_n - 0| < \epsilon$$

□

We were unable to make use of the Algebraic Limit Theorem as it assumes knowledge of both the limits of two sequences and the convergence of (a_n) was unknown.

(b) No, given that the convergence of $(a_n b_n)$ would then be dependent on the convergence of (a_n) also, ie: if $(a_n) = \{1, -1, 1, -1, 1, ...\}$ then $(a_n b_n)$ will not converge.

(c) Within the Algebraic limit theorem it is assumed that both sequences are convergent so for this case we have

$$a_n \to a \qquad\qquad b_n \to 0.$$

and given that all convergent series are bounded then this case reduces to that provided in (a) showing

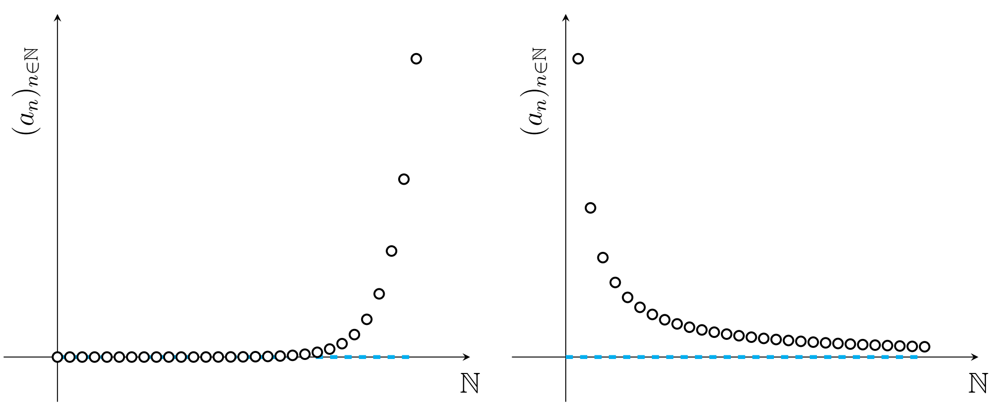
$$\lim(a_n b_n) = a0 = 0.$$

2.4 The Monotone Convergence Theorem

If there exists a wall and one runs towards it, one is undoubtedly expected to run into it !

In **Section 2.3.1** it was proved that *every convergent series is bounded* and it was noted that when reversing the argument, boundedness is a necessary condition for convergence but not a sufficient argument. Another property in conjunction to boundedness, which describes the direction in which the sequence proceeds in terms of *order*, creates a sufficient descriptor for a convergent sequence. This is known as the monotone convergence theorem.

Definition 2.4.1. A sequence is *increasing* if $a_n \leq a_{n+1} \forall n \in \mathbb{N}$.
A sequence is *decreasing* if $a_n \geq a_{n+1} \forall n \in \mathbb{N}$.
A sequence is **monotone** if it fits one of these two definitions exclusively.

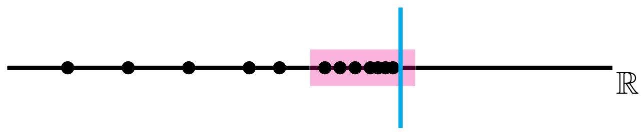


(a) Monotone Increasing (b) Monotone Decreasing

Theorem 2.4.1. ***(Monotone Convergence Theorem).*** *If a sequence is monotone and bounded, then it converges.*

Proof. The proof will follow directly by showing that if some sequence (a_n) is assumed to be monotone and bounded it must converge.
The proof is symmetric and thus applies to both increasing and decreasing sequences but let (a_n) be monotone increasing sequence for the sake of proof.
To show convergence we must find a limit and an associated ϵ-neighbourhood with some $N \in \mathbb{N}$. For $(a_n) = \{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$, given that (a_n) is a bounded above sequence in $\mathbb{R}$ by the assumption, then it follows from the Completeness Axiom that (a_n) must have a supremum.

$$s = \sup\{a_n : n \in \mathbb{N}\}.$$



Naturally, one is now guided to the notion that the supremum must be the limit, ie: the wall we're running into. Thus the proof will follow by showing that this is true.

Let $\epsilon > 0$ be arbitrary.
By nature of the supremum, there exists some point a_N such that

$$s - \epsilon < a_N$$

This implies that $\forall n \geq N$ given that the sequence is monotone increasing, $a_N \leq a_n$ giving

$$s - \epsilon < a_N < a_n.$$

But since s is a supremum by the boundedness of (a_n), then

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon.$$

$$\implies |a_n - s| < \epsilon$$

Therefore the sequence converges to s by the definition of convergence as required. □

2.4.1 Examples

An Example involving Recursive Definition using Induction This is Question 6 from Prof.Buhagair's problem sheet and Ex.2.4.2 from Abbott.

(a) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4-x_n}$ converges.

(b) Now that we know $\lim(x_n)$ exists, explain why $\lim(x_{n+1})$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) of this exercise to explicitly compute $\lim(x_n)$.

Proof. For (a) we first try to see what values the sequence pops out.

$$x_1 = 3 \tag{2.31}$$

$$x_2 = \frac{1}{4-3} = 1 \tag{2.32}$$

$$x_3 = \frac{1}{4-1} = \frac{1}{3} \tag{2.33}$$

$$\vdots \tag{2.34}$$

The sequence appears to be decreasing. This is suggestive of the use of the Monotone Convergence theorem and so we will first begin by proving that the sequence is monotone decreasing. Given that the sequence is recursively defined we take an approach by induction.

Evidently the base case holds as shown by (2.31) and (2.32).

As an inductive hypothesis let $x_n > x_{n+1}$.
Consider that by definition of the sequence,

$$x_{n+1} = \frac{1}{4 - x_n} \qquad\qquad x_{n+2} = \frac{1}{4 - x_{n+1}}$$

Now by the induction hypothesis $x_n > x_{n+1}$

$$\implies \frac{1}{4 - x_n} > \frac{1}{4 - x_{n+1}}$$

$$\therefore\ x_{n+1} > x_{n+2}$$

Therefore by the principle of induction the sequence is monotone decreasing, as required.

Awesome so we've shown that that the sequence is decreasing and that's half of what we need to show convergence, now we must show boundedness. The sequence at hand is by construction bounded above but we need the sequence to be bounded below so that the Monotone Convergence Theorem can be applied. Again an approach by induction will be taken. We claim that the sequence is bounded below by 0.

Evidently the base case holds as shown by (2.31).

As an inductive hypothesis let $x_n > 0$.
Consider that by definition of the sequence,

$$x_{n+1} = \frac{1}{4 - x_n}.$$

Now the numerator is positive and given that $3 \geq x_n > 0$ then $4 - x_n > 0$ also giving

$$\therefore\ x_{n+1} > 0.$$

Thus the sequence is bounded below by 0, as required.

All conditions required by the Monotone Convergence theorem have been met showing that (x_n) is indeed convergent. $\square$

For (b) consider that (x_{n+1}) is just (x_n) shifted by 1 term. As such, (x_{n+1}) converges to the same limit as that of (x_n).

For (c) consider the recursive formulation of the sequence

$$x_{n+1} = \frac{1}{4 - x_n}.$$

Taking the limit of both sides

$$\lim(x_{n+1}) = \lim\left(\frac{1}{4 - x_n}\right).$$

Applying the Algebraic Limit Theorem

$$\lim(x_{n+1}) = \frac{\lim(1)}{\lim(4 - x_n)}$$
$$\lim(x_{n+1}) = \frac{\lim(1)}{\lim(4) - \lim(x_n)}$$

Given that 1 and 4 are constant

$$\lim(x_{n+1}) = \frac{1}{4 - \lim(x_n)}$$
$$\lim(x_{n+1})\left(4 - \lim(x_n)\right) = 1$$
$$4\lim(x_{n+1}) - \lim(x_{n+1})\lim(x_n) = 1$$

Given that by (b) $\lim(x_n) = \lim(x_{n+1})$

$$\text{Let } \lim(x_n) = \lim(x_{n+1}) = a$$
$$\implies 4a - a^2 = 1$$
$$a^2 - 4a + 1 = 0$$

Applying completing the square

$$(a-2)^2 - (2)^2 + 1 = 0$$
$$a = 2 \pm \sqrt{3}$$

But given that both sequences are bounded above by 3 since they are decreasing then

$$\underline{\underline{\lim(x_n) = \lim(x_{n+1}) = 2 + \sqrt{3}}}$$

An Example involving Square Roots Let (a_n) be a sequence defined by $a_1 = \sqrt{2}$ and $a_n = \sqrt{a_{n-1} + 2}\ \forall n > 1$. Prove that a sequence defined in this manner is convergent.

Proof. Again we first try to see what values the sequence pops out.

$$a_1 = \sqrt{2} \tag{2.35}$$
$$a_2 = \sqrt{\sqrt{2} + 2} \tag{2.36}$$
$$a_3 = \sqrt{\sqrt{\sqrt{2} + 2} + 2} \tag{2.37}$$
$$\vdots \tag{2.38}$$

This sequence of irrational numbers appears to be increasing thus a proof of convergence by Monotone Convergence is probably the most viable route. To start let's prove that the sequence is monotone increasing by induction.

The base case clearly holds as

$$\sqrt{\sqrt{2}+2} \approx 1.848 > \sqrt{2} \approx 1.414.$$

As an induction hypothesis let $a_n < a_{n+1}$ and thus to prove that this holds throughout we must show that $a_{n+1} < a_{n+2}$.

Consider that a_{n+1} and a_{n+2} can be recursively defined such that

$$a_{n+1} = \sqrt{a_n + 2} \qquad\qquad a_{n+2} = \sqrt{a_{n+1} + 2}.$$

Given that by the induction hypothesis $a_{n+1} > a_n$ then

$$\begin{aligned} a_{n+1} + 2 &> a_n + 2 \\ \implies \sqrt{a_{n+1} + 2} &> \sqrt{a_n + 2} \\ \therefore\ a_{n+2} &> a_{n+1}. \end{aligned}$$

Thus, given that the inductive step has been satisfied, by the principle of induction (a_n) is monotone increasing.

Alternatively the sequence can be shown to be monotone increasing by considering that by definition

$$\begin{aligned} a_{n+1} = \sqrt{a_n + 2} &\geq a_n \\ \implies a_n^2 - a_n - 2 &\geq 0 \\ (a_n - 2)(a_n + 1) &\geq 0 \end{aligned}$$

But given that the sequence is non-negative then only the implication of the left bracket is valid. Thus by nature of the square root and the definition of the sequence

$$0 \leq a_n \leq 2.$$

Personally I much prefer the proof by induction.

Now, the sequence must be shown to be bounded above. It is clearly bounded below by $\sqrt{2}$ but it appears to me that the sequence can at most hopes to reach the value of 2 so let's try to prove this by induction.

Clearly for the base case this stands since

$$\sqrt{2} < 2.$$

As an inductive hypothesis allow $a_n \leq 2$ meaning that the inductive step to prove is that $a_{n+1} < 2$.

Consider that a_{n+1} can be recursively defined by the given definition of the sequence as

$$a_{n+1} = \sqrt{a_n + 2}.$$

But the induction hypothesis gives

$$\begin{aligned} a_n &\leq 2 \\ a_n + 2 &\leq 2 + 2 = 4 \\ \implies \sqrt{a_n + 2} &\leq \sqrt{4} = 2 \\ \therefore\ a_{n+1} &\leq 2 \end{aligned}$$

Thus the sequence is bounded above by 2 as required.
Therefore by the Monotone Convergence Theorem, the sequence is convergent. □

To evaluate the limit, which is intuitively 2, consider that $\lim a_{n+1} = \lim a_n$ given that (a_{n+1}) is just (a_n) displaced by one value and take limits of both sides of the defining relation

$$\begin{aligned} a_{n+1} &= \sqrt{a_n + 2} \\ a_{n+1}^2 &= a_n + 2 \\ \lim(a_{n+1}^2) &= \lim(a_n + 2) \end{aligned}$$

By the Algebraic Limit Theorem

$$\lim(a_{n+1}) \lim(a_{n+1}) = \lim a_n + \lim 2$$

Now taking $\lim(a_{n+1}) = \lim a_n = a$ and consider that (2) is a constant sequence

$$\begin{aligned} a^2 - a - 2 &= 0 \\ (a-2)(a+1) &= 0 \end{aligned}$$

But $0 \leq a_n \leq 2$

$$\therefore\ \lim(a_{n+1}) = \lim a_n = a = 2.$$

Calculating Square Roots This question may be found as problem 7 from Prof.Buhagair's Problem Sheet 2 and Ex.2.4.5 from Abbott.

Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that $x_n^2 > 2\,\forall\, n \in \mathbb{N}$, and hence that $x_n - x_{n+1} \geq 0$. Conclude that $\lim_{n\to\infty} x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

For (a) we must essentially provide a proof of convergence, the recursive nature leading one to think about using induction and the monotone convergence theorem.

Let's play around with the definition a bit and get some values out to get an intuition for what we can do.

$$x_1 = 2 \tag{2.39}$$
$$x_2 = \frac{3}{2} \tag{2.40}$$
$$x_3 = \frac{17}{12} \tag{2.41}$$
$$\vdots \tag{2.42}$$

Clearly the sequence seems to be decreasing. Now consider this alternate form of the definition;

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \tag{2.43}$$
$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \tag{2.44}$$
$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}. \tag{2.45}$$

With this reformulation in hand it becomes clear that to achieve $x_n^2 > 0$ we need to show that the sequence is bounded by 0 which makes sense for a sequence involving square terms.
Thus our criteria for the application of the monotone convergence theorem have been achieved, we must show that the sequence is non-negative and that it is monotone decreasing.

Proof. First, to prove that the sequence is bounded below by 0 we take an approach by induction.

The base case evidently holds given that $x_1 = 2 > 0$.
The inductive hypothesis is such that $x_n > 0$ and if it is to hold and the whole sequence shown to be non-negative then we must have $x_{n+1} > 0$ to satisfy the inductive step.
Consider that by definition of the sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

This expression contains only positive terms by the induction hypothesis and as a result

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) > 0.$$

Thus by the principle of induction
$$(x_n) > 0$$
showing that (x_n) is bounded by 0.
Now considering the reformulation given earlier this gives

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}.$$

By messing around with this equation, one notes that the following application lead to the desired inequality, alternatively one would use induction.

$$x_{n+1}^2 - 2 = \left(\frac{x_n^2 + 2}{2x_n}\right)^2 - 2$$
$$x_{n+1}^2 - 2 = \frac{(x_n^2 + 2)^2}{4x_n^2} - 2$$
$$x_{n+1}^2 - 2 = \frac{x_n^4 + 4x_n^2 + 4}{4x_n^2} - 2$$
$$x_{n+1}^2 - 2 = \frac{x_n^4 + 4x_n^2 + 4 - 8x_n^2}{4x_n^2}$$
$$x_{n+1}^2 - 2 = \frac{x_n^4 - 4x_n^2 + 4}{4x_n^2}$$

which is clearly factorisable to

$$x_{n+1}^2 - 2 = \frac{(x_n^2 - 2)^2}{4x_n^2}.$$

Now since this is a squared expression

$$(x_n^2 - 2)^2 > 0$$
$$x_n^2 > 2 \text{ as required.}$$

Moving on to proving that (x_n) is monotone decreasing. Consider the fact that if (x_n) is decreasing then this would mean that

$$x_n > x_{n+1} \text{ and so } x_n - x_{n+1} > 0.$$

Where as if the sequence where increasing this same difference would be negative. Now, from (2.45)

$$x_n - x_{n+1} = x_n - \frac{x_n^2 + 2}{2x_n}$$
$$x_n - x_{n+1} = \frac{2x_n^2 - x_n^2 - 2}{2x_n}$$
$$x_n - x_{n+1} = \frac{x_n^2 - 2}{2x_n}$$

Now consider that from earlier $x_n^2 > 2$ and $x_n > 0$ which means that

$$x_n - x_{n+1} = \frac{x_n^2 - 2}{2x_n} > 0$$
$$\therefore x_n - x_{n+1} > 0 \text{ as required.}$$

(x_n) has been shown to be bounded below by 0 and to be monotone decreasing, thus applying the monotone convergence theorem means that (x_n) is convergent. □

To evaluate the limit, the limit of both sides of the recursive definition are considered

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$
$$\lim(x_{n+1}) = \lim\left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)$$

By application of the algebraic limit theorem

$$\lim(x_{n+1}) = \lim\left(\frac{1}{2}\right)\lim\left(x_n + \frac{2}{x_n}\right)$$
$$\lim(x_{n+1}) = \lim\left(\frac{1}{2}\right)\lim(x_n) + \lim\left(\frac{1}{2}\right)\frac{\lim 2}{\lim x_n}.$$

But now since $\left(\frac{1}{2}\right)$ and (2) are constant sequences

$$\lim(x_{n+1}) = \frac{1}{2}\lim(x_n) + \frac{1}{\lim x_n}.$$

Given that (x_n) and x_{n+1} are the same sequence displaced by one point, they must converge to the same point and as a result let $\lim(x_{n+1}) = \lim(x_n) = a$.

$$a = \frac{1}{2}a + \frac{1}{a}$$
$$a = \frac{a^2 + 2}{2a}$$
$$2a^2 = a^2 + 2$$
$$a^2 - 2 = 0$$
$$\therefore\ a = \pm\sqrt{2}$$

But, given that the sequence is bounded below by 0

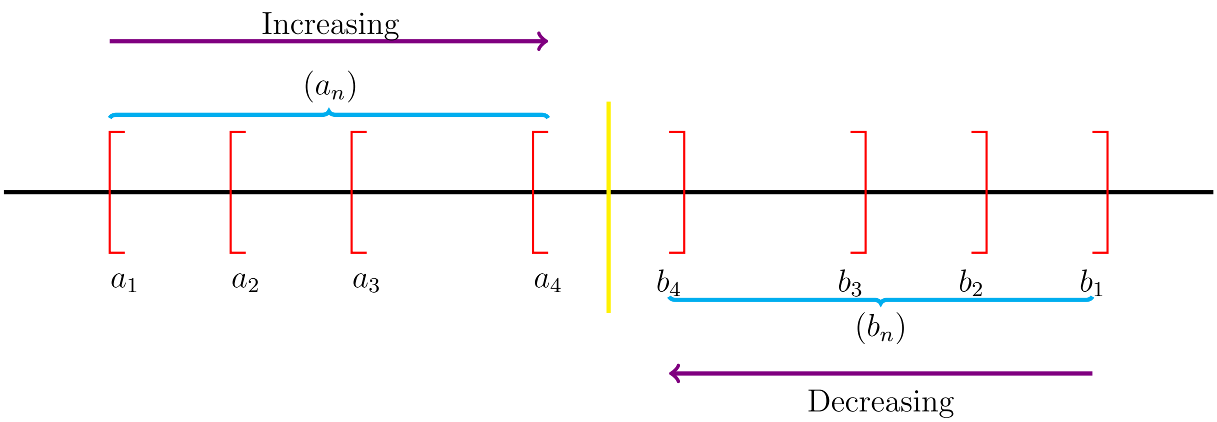
$$\lim(x_{n+1}) = \lim(x_n) = a = \sqrt{2}.$$

For b).
It must surely follow by preservation of structure that a sequence recursively defined by

$$x_{n+1} = \frac{1}{c}\left(x_n + \frac{c}{x_n}\right) > 0$$

will converge to $\sqrt{c}$.

The Stronger Nested Interval Property



Earlier when introducing the Nested Interval Property we thought of a sequence of nested intervals and using this idea of nesting came to the conclusion that their intersection is non-empty.
In the Stronger restatement of this property we formalise it using the notion of a limit to show that the size of the interval is decreasing such that it approaches 0 at infinity and as a result the intersection of all the intervals contains only one point. This is what make this formulation *stronger*, where as earlier we showed that the intersection of the nested intervals is non-empty, we now show that contains only one point.

Theorem 2.4.2. *Let $(I_n = [a_n, b_n])_{n\in\mathbb{N}}$ be a decreasing sequence of non-empty closed intervals : $I_{n+1} \subseteq I_n$, with the property that*

$$\lim_{n\to\infty}(b_n - a_n) = 0.$$

Then there is a unique point which belongs to all the intervals.

Proof. Starting with **Theorem 1.1.1.**

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

We want to show that this intersection is occupied by one unique point.

First consider the left bracket of each interval and the right bracket of each interval. Individually both of these bracket groupings form sequences, let's call them (a_n) and (b_n). It is clear that by construction of nested intervals that (a_n) is increasing and bounded above by some b_1 whilst (b_n) is decreasing and bounded below by some a_1. Thus, by the Monotone Convergence Theorem both sequences are convergent.

$$\lim_{n\to\infty} a_n = a \qquad\qquad \lim_{n\to\infty} b_n = b$$

Now by design $a_n \leq b_n$ and so by the Order limit Theorem $a \leq b$. But, by the premise $\lim_{n\to\infty}(b_n - a_n) = 0$ and by the Algebraic Limit Theorem we may manipulate this given that we have proven the convergence of both (a_n) and (b_n).

$$\lim_{n\to\infty}(b_n - a_n) = 0$$
$$\lim_{n\to\infty}(b_n) - \lim_{n\to\infty}(a_n) = 0$$
$$\implies \lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(b_n)$$
$$\therefore\ a = b.$$

And so giving

$$a_n \leq a = b \leq b_n \,\forall\, n \in \mathbb{N}$$
$$\implies a = b \in \bigcap_{n=1}^{\infty} I_n.$$

Thus showing that both sequences converge to the same point in the intersection of all the nested intervals, but is it the only thing in this intersection ?
Say there exists some other point in the intersection such that

$$c \in \bigcap_{n=1}^{\infty} I_n.$$

This infers that

$$a_n \leq c \leq b_n$$

but given that $\lim a_n = \lim b_n$ then by the Sandwich Lemma

$$\lim y = y = \lim a_n = \lim b_n \in \bigcap_{n=1}^{\infty} I_n.$$

Proving, that intersection of the nested intervals is occupied by one unique point. $\square$

Had I had the time $\limsup a_n$ **and** $\liminf a_n$

2.5 Subsequences and the Bolzano - Weierstrass Theorem

2.5.1 Subsequences

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < ...$ be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

is called a subsequence of (a_n) denoted (a_{n_j}), where $j \in \mathbb{N}$ indexes the subsequence.

The notion of a subsequence stems from the fact that smaller increasing sequences can be found in increasing sequences. The example from page 47 involving the shuffling of sequences is suggestive of this concept;

$$\longrightarrow$$
$$\underline{a_1}\,\underline{a_2}\,\underline{a_3}\,\underline{a_4}\ldots\ldots$$
$$\underline{a_1}\quad\ \underline{a_3}\qquad\ldots\ldots$$
$$\underline{a_2}\quad\ \underline{a_4}\ldots\ldots$$

Consider that from above, with the new definition in mind, the sequence (a_n) has two subsequences

$$(a_{1_1}, a_{3_2}, a_{5_3}, \ldots) \qquad\qquad (a_{2_1}, a_{4_2}, a_{6_3}, \ldots)$$

Legitimate subsequences are ones which occur within the same order as that of the over arching sequence and do not feature repeated elements from the over arching sequence meaning that

$$(\ldots, a_{5_1}, a_{3_2}, a_{1_3}) \qquad\qquad (a_{2_1}, a_{2_2}, a_{4_3}, a_{6_4}, \ldots)$$

are not legitimate subsequences.

Theorem 2.5.1. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

This was alluded to in the example on page 47.

On a conceptual level think about it like this;*if there is a path with a wall at the end, whether you walk down the path or jump along the path you will still end up at the wall.*

Mathematically what we're saying here is that if we have some sequence (a_n) with subsequence (a_{n_k}) and $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} a_{n_k} = a$ also.

Proof. Given $\lim_{n\to\infty} a_n = a$ we show that $\lim_{k\to\infty} a_{n_k} = a$ also.
By definition of convergence

$$\exists\ N \in \mathbb{N} : \forall\, n \geq N,\ |a_n - a| < \epsilon.$$

Now consider that given the fact that subsequences are ordered in the same way as the sequence that the index of the sequence is always greater or equal to the index of the subsequence

$$n_k \geq k.$$

Thus with k reaching the ϵ-neighbourhood

$$k \geq N \implies n_k \geq N.$$

$$\therefore\ |a_{n_k} - a| < \epsilon.$$

$\square$

The contrapositive of **Theorem 2.5.1** provides a criterion for **Divergence** such that *sequences which diverge have subsequences that converge to different limits.* Looking back at the example of a divergent sequence from **Section 2.2.4.** this becomes even more evident;

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, ...\right).$$

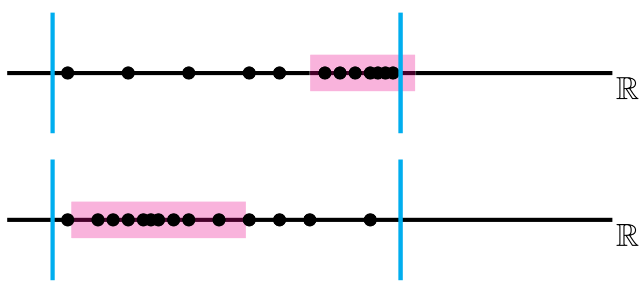
the clearest two subsequences with differing limits from this sequence are

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots\right) \qquad \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right)$$

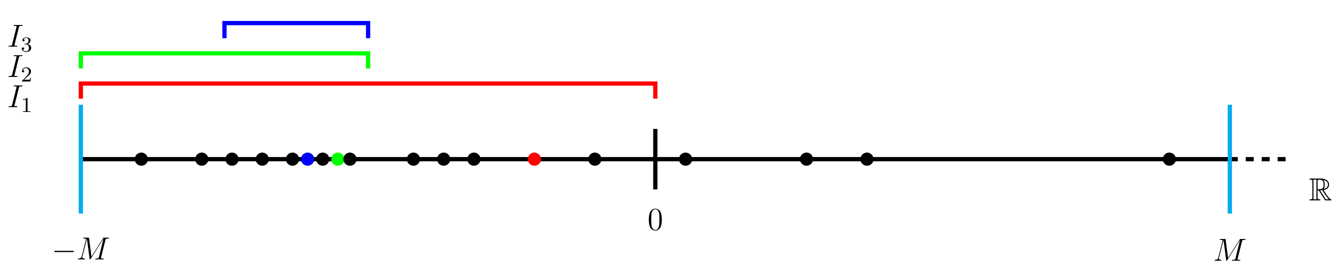
which clearly converge to $-\frac{1}{5}$ and $\frac{1}{5}$.
Again this is intuitive, having multiple walls we can't definitively know which one we're going to hit.

2.5.2 The Bolzano-Weierstrass Theorem

It may at times be difficult to discern a convergent subsequence within a sequence and thus what the Bolzano-Weierstrass Theorem provides is a means of proving that a sequence does in fact contain a convergent subsequence.
Theorem 2.5.2. *Every bounded sequence contains a convergent subsequence.*



From a conceptual stand point we have something that's living exclusively between two walls and what we are claiming is that at least some part of this thing must hit one of these two walls. In particular if there is some monotone subsequence then it is straightforward to see that it will converge.
Such a subsequence will be convergent by the monotone convergence theorem by being monotone increasing and bounded above or monotone decreasing and bounded below thus we must split or sequence in half to begin looking for a monotone subsequence. This notion of halving or bisecting will be the cornerstone of the presented proof to construct a convergent subsequence making use of the strong nested interval property.



Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M \,\forall\, n \in \mathbb{N}$. We thus have the closed interval $[-M, M]$, now splitting that interval we have $[-M, 0]$ and $[0, M]$ and choosing one of these closed intervals such that it contains an infinite number of points of (a_n) and let's label this section I_1 select $a_{n_1} \in (a_n)$ which is in I_1 also.
Now, repeating this process; I_1 is split into two segments of equal length and once more the closed interval with infinitely many points of (a_n) is labelled I_2, selecting $a_{n_2} \in I_2 \,:\, n_2 > n_1$.
In general we repeat this, constructing the k-th closed interval I_k by splitting I_{k-1} into two and labelling the closed interval with infinitely many points of (a_n) as I_k. a_{n_k} is selected such that $n_k > n_{k-1} > \cdots > n_2 > n_1$ such that $a_{n_k} \in I_k$.
By construction, we now wish to show that (a_{n_k}) is a convergent subsequence, so if it is, what's it's limit ? Well, each point in (a_{n_k}) comes from being associated to a closed interval and it is evident that

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{k-1} \supset I_k.$$

From **Theorem 2.4.2** we understand that by the stronger nested interval property for a decreasing sequence of non-empty closed intervals such that the size of the interval approaches 0 at infinity, there is a unique point in the intersection of all the intervals. It is evident that some $x \in \bigcap_{k\in\mathbb{N}} I_k$ would be the limit of (a_{n_k}) so let's prove it.

Let $\epsilon > 0$. By construction the length of I_k is $\frac{M}{2^{k-1}}$ such that I_1 was of size M and $l(I_2) = \frac{M}{2^1}$. Now, taking the fact that $\frac{1}{n}$, is convergent to 0, as shown in **Example 1** of **Section 2.2.3.**, and using **Theorem 2.5.1** it is evident that; $\frac{M}{2^{k-1}}$ being a subsequence of $\frac{1}{n}$ also

$$l(I_k) = \frac{M}{2^{k-1}} \to 0.$$

Thus the series of nested intervals now has shown to conform to do conditions required by the stronger nested interval property and

$$\bigcap_{k\in\mathbb{N}} I_k = \{x\}.$$

So we've shown that there is an x in the intersection of all the intervals, all is left is to prove it is the limit of (a_{n_k}).
Choosing $N \in \mathbb{N}$ such that the length of I_k is smaller than ϵ then given that $x, a_{n_k}, \in I_k$ then $\forall\, k > N$

$$|a_{n_k} - x| < \epsilon.$$

□

To recap; we started with a bounded sequence and halfed it repeatedly confining elements of some subsequence to form a monotone subsequence. We then showed that these halves were nested intervals and given that by construction the size of these halves was approaching 0 then there was a point common to all the halves, by the strong nested interval property. We then showed that this point was indeed the limit of the subsequence by asserting that the size of size the k-th interval was to be smaller than the size of the epsilon neighbourhood.

2.5.3 Examples and Exercises

Exercise 2.5.3 This is listed as given on Page 58 of Abbott and as Problem 8 on Prof.Buhagair's Problem Sheet 2.
Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(b) A monotone sequence that diverges but has a convergent subsequence.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

(d) An unbounded sequence with a convergent subsequence.

(e) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

For (a). Consider that by **Theorem 2.5.1** such a sequence is divergent. Now we showed at the beginning of **Section 2.2.3.** that the harmonic sequence converges to 0 so that's definitely going to be one of my subsequences. Now to converge to 1 consider a sequence of the $1 + \frac{1}{n}$ of the harmonic sequence giving terms like $\frac{n+1}{n}$.
The choice of this subsequence is sensible by the Algebraic limit theorem as we are adding the constant series of 1 to the harmonic sequence such that

$$\lim\left(1_n + \frac{1}{n}\right) = \lim(1_n) + \lim\left(\frac{1}{n}\right) = 1 + 0 = 1.$$

This would make sense as a decreasing sequence so for it to fit in with an all increasing sequence take $\frac{n-1}{n}$ terms. Thus the example will look something like this

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \ldots, \frac{1}{n}, \ldots, \frac{n-1}{n}, \ldots\right).$$

For (b). By the Monotone Convergence theorem a monotone sequence that diverges is unbounded and given that the Bolzano-Weierstrass Theorem provides that convergent subsequences are found in bounded sequences, a convergent subsequence cannot exist in an unbounded sequence.

For (c). This is possible constructing a sequence with constant sequences of each value of a point in the harmonic sequence

$$\left(1, 1, 1 \ldots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right).$$

It is good to note that a subsequence of this sequence is the harmonic sequence itself, many times over in fact, and thus each of these would converge to 0.
Correction ! : Infinite sequences can only have infinite subsequences. Perhaps this is because we have removed ourselves from the definition of a sequence but recall that a

sequence is a function whose domain is $\mathbb{N}$, which is infinite by nature.
Thus both sequences and their subsequences must be infinite and in the answer given above the subsequences are finite. To construct the desired infinite subsequences we employ a diagonal argument *a la* Cantor in the following way

$$
\begin{array}{ccccccc}
1 & \longrightarrow & 1 & & 1 & \longrightarrow & \dots \\
 & \swarrow & & \nearrow & & \swarrow & \\
\frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \dots \\
 & \nearrow & & \swarrow & & \nearrow & \\
\frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & & \dots \\
 & \swarrow & & \nearrow & & \swarrow & \\
\frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & & \dots \\
 & \nearrow & & \swarrow & & \nearrow & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

generating subsequences which are infinite in nature and a sequence of form

$$\left(1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} ..\right).$$

Refer to **Section 1.2.7.** for a deeper discussion.

For (d). An unbounded sequence can have an bounded convergent subsequence. Consider the increasing n sequence and inserting a 1 after every point gives a constant subsequence of 1s which converges to 1.

$$(1, 1, 2, 1, 3, 1, 4, 1, 5, 1, \dots).$$

For (e). Consider that by the Bolzano-Weierstrass Theorem a bounded subsequence itself must contain convergent subsequences, or convergent subsubsequences if you like, relative to the over-arching sequence. Thus it is impossible to have a bounded subsequence without convergent subsequences.

Exercise 2.5.4. This is listed as given on page 58 of Abbott and as Question 9 from Prof.Buhagair's Problem Sheet 2.
Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Instantly this is reminiscent of the example of sequence shuffling from page 47 of these notes. Here we are essentially tasked with proving a more general form of that case, for which I think we shall employ the Bolzano-Weierstrass Theorem to prove.
This being said a direct route for the proof is not evident but the statement seems obvious so an approach by contradiction will be taken.

Proof. Assume (a_n) is a bounded sequence which does not converge to a. This means that negating the definition of convergence, for all $N \in \mathbb{N}$ there exists some $n \geq N$ for which

$$|a - a_n| \geq \epsilon \, : \, \epsilon > 0.$$

Now we construct a subsequence that never enters the the ϵ-neighbourhood on the basis of the above, such that the first point in the subsequence would be

$$|a - a_{n_1}| \geq \epsilon$$

and going on such that

$$n_1 < n_2 < \cdots < n_j < n_{j+1} < \ldots$$

which is afforded by the fact that for all $N \in N$ we can find some $n \geq N$ which is not in the neighbourhood from the negation of the definition of convergence. This infers that (a_{n_j}) exists exclusively outside of $V_\epsilon(a)$ and by the fact that (a_n) is bounded, (a_{n_j}) is bounded also. Thus applying the Bolzano-Weierstrass theorem (a_{n_j}) should have some convergent subsequence $(a_{n_{j_k}})$. Now, $(a_{n_{j_k}})$ being a subsequence of (a_{n_j}) exists outside of $V_\epsilon(a)$ and thus a subsequence of (a_n) does not converge to a ※. □

Example 2.5.3 & Exercise 2.5.5. Show that (b^n) is a null sequence for $-1 < b < 1$.

This proof can be split into 3 distinct cases depending on the nature of the sequence within these 3 sections;

(a) $0 < b < 1$.

(b) $b = 0$.

(c) $-1 < b < 0$.

Proof. For (a). We have a monotone decreasing sequence that is bounded below by 0;

$$b > b^2 > \cdots > 0.$$

Thus by the Monotone convergence theorem we have than (b^n) to some l such that $b > l \geq 0$. Now in order to determine l consider that the subsequence formed by all the even exponents, (b^{2n}) should have the same limit as the over arching sequence for convergence by **Theorem 2.5.1**. But consider that by the Algebraic Limit Theorem

$$\lim(b^{2n}) = \lim(b_n)\lim(b_n) = l \cdot l = l^2.$$

But $\lim(b^{2n}) = \lim(b^n)$ by **Theorem 2.5.1.** thus the only number fitting this description is 0.

$$\implies \lim(b_n) = 0, \text{ for } 0 < b < 1.$$

For (b). We have constant sequence such that

$$0 = 0^2 = 0^3 = \cdots = 0^n = \ldots.$$

As a result it is evident that $\lim(0^n) = 0$.

For (c). We have the sequence from (a) but instead the input is negative and as such the sequence can be written $(-b^n)$. With this in hand the Algebraic Limit Theorem can be applied giving

$$\lim(-b^n) = \lim(-1^n)\lim(b^n) = (-1)(0) = 0.$$

Thus we have shown that for all the given range, $-1 < b < 1$, (b^n), is convergent. $\square$

2.5.4 The Cauchy Criterion

Definition 2.5.2. A sequence (a_n) is called a Cauchy sequence if for ever $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

In the lines to come it will be shown that this formalism is **equivalent to convergence** but approaches from a different view, that of proximity of points. Where within the definition of convergence we concerned ourselves with showing that the points of a series all come to be very close to some singular point known as a limit. Within the approach of Cauchy sequences **we do not need to know what the limit is** (as within the monotone convergence theorem) to know that a sequence is convergent. Instead we show the points of some sequence get very close to each other within the tail of some sequence and that shows convergence.

Theorem 2.5.3. *Every convergent sequence is a Cauchy sequence.*

This is the forward implication of the equivalence of convergence and a sequence being Cauchy. As such we will start with the definition of convergence and directly show that it is mathematically equivalent to to the definition of a Cauchy sequence.

Proof. Let (a_n) be a sequence converging to some limit a and let $\epsilon > 0$ be arbitrary. Now by the definition of convergence this means that there exists some $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$|a_n - a| < \frac{\epsilon}{2} \qquad |a_m - a| < \frac{\epsilon}{2}.$$

Now we are concerned with the proximity of these two points and want to show that this is less than ϵ for the sequence to be Cauchy. Consider that applying a midpoint trick to their difference

$$|a_n - a_m| = |a_n - a + a - a_m|.$$

By the triangle inequality we have that

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a|.$$

And by the presumed convergence

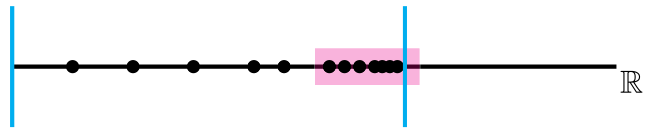
$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

$$\implies |a_n - a_m| < \frac{\epsilon}{2} \text{ as required.}$$

$\square$

Now to show the backwards implication we will make use of the Bolzano-Weierstrass Theorem but first we must show that Cauchy sequences are bounded and that convergent subsequences of Cauchy sequences, converge to the same limit as that of the sequence.

Lemma 2.5.4. *Cauchy sequences are bounded.*



Intuitively this makes sense. If the points of a sequence are going to get increasingly close to each other within the tail of some sequence then the points cannot seek to move forward past this clumping point. If a such a sequence were not bounded then there would be no clumping and points would 'get away' from this proximity point.

Proof. Let (a_n) be a Cauchy sequence and for conceptual tangibility let $\epsilon = 1$.
Now by the definition of Cauchy there exists some $N \in \mathbb{N}$ such that $n, m \geq N$ imply that

$$|a_n - a_m| < 1.$$

Thus N defines a region were all points are closer than 1 to each other and as such any point in this region should be smaller than some $|x_N| + 1$

$$\implies |x_n| < |x_N| + 1 \, \forall n \geq N.$$

Which means that the (a_n) would be bounded by

$$M = \max\{|x_1|, |x_2|, \dots, |x_N|, \dots, |x_n|, \dots, |x_N| + 1\}.$$

□

Lemma 2.5.5. *A Cauchy sequence with a convergent subsequence converges to the same limit.*

Let (a_n) be a Cauchy sequence and $(a_{n_k}) \to a$ be a convergent subsequence of (a_n).
Let $\epsilon > 0$ be arbitrary. By the definition of convergence we wish to show that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|a_n - a| < \epsilon.$$

Now we now that by the premise (a_n) is a Cauchy Sequence and as such there exists some $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies

$$|a_n - a_m| < \frac{\epsilon}{2}.$$

We also have that for some $N_2 \in \mathbb{N}$, for all $n_k \geq N_2$ we have

$$|a_{n_k} - a| < \frac{\epsilon}{2}.$$

Consider that $|a_n - a|$ can be reformulated by the application of a midpoint trick

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a|.$$

And by the triangle inequality this gives

$$|a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a|.$$

Choosing $N = \max\{N_1, N_2\}$ and letting $n, n_{n_k} \geq N$ gives

$$|a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\therefore\ |a_n - a| < \epsilon.$$

Theorem 2.5.6. *Every Cauchy sequence is a convergent sequence.*

This is the backwards implication of the equivalence of convergence and Cauchy sequences. We will use the two preceeding Lemmas in conjunction withthe Bolzano-Weierstrass theorem to show the backwards implication of equivalence.

Proof. Let (a_n) be a Cauchy sequence. As a result of **Lemma 2.5.4.**, (a_n) is bounded and so by the Bolzano-Weierstrass Theorem, (a_n) must have some convergent subsequence, say (a_{n_k}).
By **Lemma 2.5.5.** we know that the Cauchy sequence converges to the same limit of its convergent subsequence and thus we have shown that a sequence being Cauchy implies that it is convergent. □

Remark 2.5.7. *All the theorems which we proved in this section are in some way linkable to the Axiom of Completeness and all in fact assert the completeness of* $\mathbb{R}$.

$$AoC \implies \begin{cases} NIP \\ MCT \end{cases} \implies BW \implies CC.$$

2.5.5 Examples and Exercises

Exercise 2.6.1. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A monotone sequence that is not Cauchy.

(c) A Cauchy sequence with a divergent subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

For (a). Cauchy Sequences are based on a criterion of distance between points and have no implications on the order within a sequence. Consider in fact that an alternating sequence may in fact be Cauchy

$$(1, -1/2, 1/3, -1/4, 1/5)$$

and alternating series by nature have no consistent order and are not monotone.

For (b). A monotone sequence that is not bounded is not convergent by the monotone convergence theorem and as such, not Cauchy.

$$(1, 2, 3, 4, 5, 6, \dots).$$

For (c). A subsequence of a Cauchy sequence is convergent and so cannot be divergent. This is not possible.

For (d). This is possible such that unbounded sequences can have areas for which the points have decreasing distance between them for some region.

$$(1, 1, 2, 1/2, 3, 1/3, 4, 1/4, \dots).$$

Exercise 2.6.4. Assume (a_n) and (b_n) are Cauchy sequences. Use a triangle inequality to prove $c_n = |a_n - b_n|$ is Cauchy.

Proof. Firstly, a good starting point is always to examine what we want to end up with which in this case would be a Cauchy (c_n) meaning that we want to show that for some choice of $N \in \mathbb{N}$, for $n, m \geq N$ and an arbitrary choice of $\epsilon > 0$, we will have

$$|c_n - c_m| < \epsilon. \tag{2.46}$$

Now what we are certain of possessing by the premise is that (a_n) and (b_n) are both Cauchy, giving

$$\exists\, N_1 \in \mathbb{N};\, n, m \geq N_1 \text{ gives } |a_n - a_m| < \frac{\epsilon}{2} \text{ and also } \exists\, N_2 \in \mathbb{N};\, n, m \geq N_2 \text{ gives } |b_n - b_m| < \frac{\epsilon}{2}.$$

Now consider that by the premise that $c_n = |a_n - b_n|$ we can rewrite (2.46) as follows

$$|c_n - c_m| = |a_n - b_n - (a_m - b_m)| = |a_n - a_m + b_m - b_n|.$$

And by application of the triangle inequality we have

$$|c_n - c_m| = |a_n - a_m + b_m - b_n| \leq |a_n - a_m| + |b_m - b_n|.$$

And choosing $N = \max\{N_1, N_2\}$, taking $n, m \geq N$ we get

$$|c_n - c_m| \leq |a_n - a_m| + |b_m - b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\therefore\ |c_n - c_m| < \epsilon.$$

Showing that (c_n) is Cauchy by definition. □

2.5.6 Some General Examples of Convergence Proofs

Exercise - Proofs Of Convergence Prove that the following sequences have these limits

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

(b) $\lim_{n\to\infty} \sqrt[n]{p} = 1$ for $p > 0$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

For most if not all of these convergence proofs **The Appraoch** discussed in **Section 2.2.3.** will be the way forward, particularly given that we know the limit we are looking for.
Had we not, we would first need to prove convergence in a general sense and then find the limit through some manipulation using the Algebraic Limit Theorem.

For (a). Taking the approach we first examine where we we want to end up which would be given that the sequence is null then for some arbitrary choice of $\epsilon > 0$

$$\left|\frac{1}{n^p}\right| < \epsilon.$$

Which would mean that this ϵ-neighbourhood would have some starting point $N \in N$ at which

$$\left|\frac{1}{N^p}\right| < \epsilon.$$

With this we can elucidate the choice for N, making it the subject

$$N > \frac{1}{\sqrt[p]{\epsilon}}.$$

Having found this we can start the formal proof.

Proof. Let $\epsilon > 0$ be arbitrary.
Choose $N \in \mathbb{N}$ satisfying

$$\frac{1}{\sqrt[p]{\epsilon}} < N$$

which is viable by the Archimedean property. So, in affirmation let $n \geq N$ giving

$$\frac{1}{\sqrt[p]{\epsilon}} < n$$

$$\implies \frac{1}{n^p} < \epsilon$$

$$\therefore \left|\frac{1}{n^p} - 0\right| < \epsilon.$$

□

For (b). Firstly we will employ a proof by cases because the nature of the order of the sequence changes depending on the choice of p.

For $p > 1$ we have a decreasing sequence

$$p^{\frac{1}{1}} < p^{\frac{1}{2}} < \cdots < p^{\frac{1}{n}} < \ldots.$$

Whilst for $0 < p < 1$ we have an increasing sequence, for example

$$\frac{1}{2}^{\frac{1}{1}} > \frac{1}{2}^{\frac{1}{2}} > \frac{1}{2}^{\frac{1}{3}} > \ldots$$

which is approximately

$$0.5 > 0.7071\cdots > 0.7937\cdots > \ldots.$$

Proof. So we will split this proof into three cases

(i) $p > 1$.

(ii) $p = 1$.

(iii) $1 > p > 0$.

For (i). Admittedly, proving that $(\sqrt[n]{p})$ converges to 1 is quite a challenge for $p > 1$. I first attempted a direct proof, trying to manipulate ϵ using logarithms but this is messy. Alternatively lets construct a new sequence out of $(\sqrt[n]{p})$ subtracting 1 from each term of the sequence giving $(a_n) = (\sqrt[n]{p} - 1) > 0$ meaning that now we instead have to show that this converges to 0 which through the algebraic limit theorem will sneakily allow us to show that $(\sqrt[n]{p})$ converges to 1 nonetheless.

Consider that by construction

$$\begin{aligned} \sqrt[n]{p} &= a_n + 1 \\ \implies p &= (1 + a_n)^n. \end{aligned}$$

Now considering the first two terms of the binomial expansion

$$1 + na_n \leq (1 + a_n)^n = p.$$

By making a_n subject

$$\implies 0 < a_n < \frac{p-1}{n}.$$

Now considering the limits of the edge terms of the inequality we have a constant sequence of 0s which must trivially be a null sequence and on the left a sequence which is a linear combination of the harmonic sequence which by the Algebraic Limit Theorem will give us

$$\lim_{n\to\infty} \left(\frac{p-1}{n}\right) = \lim_{n\to\infty} (p-1) \lim_{n\to\infty} \left(\frac{1}{n}\right).$$

Now the harmonic sequence was proven to be null by the Archimedean Property giving

$$\lim_{n\to\infty}\left(\frac{p-1}{n}\right) = (p-1)0 = 0$$

Thus applying the sandwhich lemma we have

$$\lim_{n\to\infty} a_n = 0$$

and so

$$\begin{aligned}
\lim_{n\to\infty} (\sqrt[n]{p} - 1) = 0 \\
\lim_{n\to\infty} \sqrt[n]{p} + \lim_{n\to\infty}(-1) = 0 \\
\lim_{n\to\infty} \sqrt[n]{p} - 1 = 0 \\
\lim_{n\to\infty} \sqrt[n]{p} = 1
\end{aligned}$$

For (ii). We have a constant sequence of 1s which is evidently convergent to 1.

For (iii). Consider that for $0 < p < 1$ the reciprocal of such a p will be greater than 1, for example;

$$\frac{1}{\frac{1}{4}} = 4$$

and so we have $\frac{1}{p} > 1$ for $0 < p < 1$. This means that extending the square root to the numerator, this case will reduce back to the first such that

$$\sqrt[n]{\frac{1}{\frac{1}{4}}} = \sqrt[n]{4}.$$

Thus for $0 < p < 1$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{p}} = 1.$$

□

For (c).

Proof. We apply the same reasoning as in (b)(i) constructing a sequence a_n and considering the terms of its binomial expansion this time taking up to the third term.
Let $a_n = \sqrt[n]{n} - 1$.

$$\implies n = (a_n + 1)^n$$

and by the binomial expansion

$$\frac{n(n-1)}{2}a_n^2 \leq (a_n + 1)^n = 1.$$

Taking a_n to be subject

$$0 < a_n \leq \sqrt{\frac{2}{n-1}}\,\forall\, n \leq 2.$$

Now we must prove that $(\sqrt{\frac{2}{n-1}})$ is a null sequence.
We wish to end up with the following for an arbitrary choice of ϵ

$$\left|\sqrt{\frac{2}{n-1}} - 0\right| < \epsilon.$$

And so there is some edge $N \in \mathbb{N}$ for this ϵ-neighbourhood which will be of form

$$\left|\sqrt{\frac{2}{N-1}}\right| < \epsilon$$

$$\implies \left|\frac{2}{\epsilon^2} + 1\right| < N.$$

This should be an adequate choice of N so in affirmation let $\forall\, n \geq N$ giving

$$\implies \left|\frac{2}{\epsilon^2} + 1\right| < n$$

$$\left|\sqrt{\frac{2}{n-1}} - 0\right| < \epsilon$$

thus we have proven that this sequence is null and thus by the sandwhich lemma we have that a_n is null also which by the algebraic limit theorem, as in (b), gives that $(\sqrt[n]{n})$ converges to 1. $\square$

Chapter 3

Infinite Series & Tests for Convergence

3.1 Series

One is advised to read up on Zeno's Paradox before embarking upon this Chapter so as to appreciate the depth of the mathematics to be discussed.

Ask yourself; if your hands are apart and you move them closer to each other to a point where they are half as apart as they were and repeat this process infinitely many times,will your hands ever touch ?

3.1.1 Preliminaries

Definition 3.1.1. Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + \dots$$

An infinite series is given by considering the sum of a sequence and as such we are in some sense see the accumulation of the value of a sequence.
Definition 3.1.2. Infinite series have a corresponding *sequence of partial sums* (s_m) which are like finite chunks of an infinite series.

$$s_m = b_1 + b_2 + b_3 + \dots + b_m.$$

Expository Example 1 Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Given that we are dealing with a squared variable then all the terms of the series are positive and some partial some will be of form

$$s_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}.$$

Given that it is positive, then some sequence of partial sums will be increasing and calling on some intuition from the preceding chapter then to show that this sequence of partial sums is convergent we must show that it is also bounded. This reasoning coming from the Monotone Convergence Theorem.
To find this upper bound let's reformulate the sequence

$$\begin{aligned} s_m &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \cdots + \frac{1}{m\cdot m} \\ s_m &< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \ldots \frac{1}{m\cdot(m-1)} \\ &= 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \ldots \frac{1}{m\cdot(m-1)} \end{aligned}$$

Now by partial fractions this gives

$$= 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots \frac{1}{m-1} - \frac{1}{m}.$$

Evidently all terms except the 1s at the start and the final term cancel out to give

$$\begin{gathered} = 1 + 1 - \frac{1}{m} \\ \therefore\ s_m < 2. \end{gathered}$$

And so by the monotone convergence theorem the sequence of partial sums converges which means that the series converges also.
Definition 3.1.3. The convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n). Specifically, the statement

$$\sum_{k=1}^{\infty} = A \iff \lim s_n = A.$$

Remark 3.1.1. *The sequence of terms of an infinite series is directly* $(a_1, a_2, a_3, \ldots)$ *whilst the sequence of partial sums* $(s_1, s_2, s_4, \ldots)$ *is a sequence of sums of terms up to a given point in the series such that* $s_n = a_1 + a_2 + \cdots + a_n$*.*
For convergence of series it is the convergence of the sequence of partial sums which we must show.
Convergence of a series is also sometimes referred to as Summability.

Expository Example 2 - Multiplication is a No No Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Again our intention is to show convergence and again we have a a sequence of positive terms which gives an increasing sequence of partial sums, thus all we must do is show that there is some bound and we proceed in a similar manner as before;

$$s_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}.$$

Now intuitively we may think that since $\frac{1}{n^2} = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ was shown to be bounded that we must have that $\sum_{n=1}^{\infty} \frac{1}{n}$ to be bounded in a situation the recalls the Algebraic Limit Theorem, but this is not so ! Indeed 2 is not even a bound

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12} > 2.$$

So for $s_{2^2} > 1 + 2(\frac{1}{2})$ and also for $s_8 > 2\frac{1}{2}$ leading to the generalisation

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} \cdots \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} \cdots \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \cdots + \frac{1}{2} \\
\implies s_{2^k} &> 1 + k\left(\frac{1}{2}\right).
\end{aligned}$$

As a result the sequence of partial sums is monotone increasing but unbounded and thus the harmonic series is divergent.
This example will be of importance when the Cauchy Condensation Test is introduced.
This shows us that the product of divergent series may yield convergent series and most importantly that the multiplication of series is weird in terms of convergence so be weary.
Let's formalise some properties of infinite series then to see what we can and can't do.

Algebraic Limit Theorem for Series
Theorem 3.1.2. *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*

(i) $\displaystyle\sum_{k=1}^{\infty} ca_k = cA \, \forall c \in \mathbb{R}.$

(ii) $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = A + B.$

Proof. For (i);
By definition of convergence of series we are lead to investigate the sequence of partial sums such that for $\sum_{k=1}^{\infty} a_k$ we have

$$s_m = a_1 + a_2 + a_3 + \cdots + a_m$$

such that

$$\lim s_m = A.$$

Now consider that $\sum_{k=1}^{\infty} ca_k$ would have a sequence of partial sums of the form

$$t_m = ca_1 + ca_2 + ca_3 + \cdots + ca_m = cs_m.$$

So by the algebraic limit theorem for sequence we have that

$$\lim(t_m) = \lim(cs_m) = c \lim s_m = cA.$$

And by the definition of series convergence

$$\sum_{k=1}^{\infty} ca_k = \lim(t_m) = cA \text{ as required.}$$

For (ii); Consider the series $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ and name their sequences of partial sums s_m and v_m giving

$$s_m = a_1 + a_2 + \cdots + a_m \qquad\qquad v_m = b_1 + b_2 + \cdots + b_m$$

thus we can have

$$c_m = s_m + v_m + (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_m + b_m)$$

which by the algebraic limit theorem for sequences will give

$$\lim(c_m) = \lim(s_m + v_m) = \lim s_m + \lim v_m = A + B.$$

And by the definition of convergence of series

$$\sum_{k=1}^{\infty} a_k + b_k = \lim(c_m) = A + B \text{ as required.}$$

□

To summarise, what we have is that infinite addition satisfies the distributive property in (i) and in (ii) series may be added as sequences. Things get messy for products due to commutativity and we'll see that later on.

Cauchy Criterion for Series

Theorem 3.1.3. *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon < 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Given that we have defined the summability of some series on basis of the convergence of its sequence of partial sums then we can reformulate this by saying the the sequence of partial sums must be Cauchy for a series to be summable, using the equivalence of Cauchy and Convergent from **Theorem 2.5.6.**

Proof. For a sequence (a_n) to be summable we have by definition that the sequence of its partial sums s_n is to be convergent and so by **Theorem 2.5.6.** Cauchy. This means that for some arbitrary $\epsilon > 0$ and for some $N \in \mathbb{N}$ such that $n > m \geq N$ we have

$$|s_n - s_m| < \epsilon.$$

And by definition of partial sums we have

$$s_n = |a_1 + a_2 + a_3 + \cdots + a_n| \qquad s_m = |a_1 + a_2 + a_3 + \cdots + a_m|$$

$$\because n > m$$

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n|.$$

And by the definition of Cauchy sequences as stated before we have

$$|a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n| < \epsilon.$$

$\square$

Corollary 3.1.3.1. *If the series $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \to 0$.*

Proof. By definition of the convergence of a series we must have a convergent sequence of partial sums also.

Consider the fact that a_n can be reformulated as the linear combination of some partial sums

$$s_n = a_1 + a_2 + a_3 + \dots a_n \qquad s_{n-1} = a_1 + a_2 + a_3 + \dots a_{n-1} \tag{3.1}$$

$$\implies a_n = s_n - s_{n-1}.$$

At this point the proof can follow in one two ways using one of the two equivalent definition for convergence the Cauchy criterion or convergence itself. We will show both in this order.

Given that $sum_{n=1}^{\infty} a_n$ is convergent we have that the sequence of partial sums must be Cauchy and so for some $N \in \mathbb{N}$ and $n > m \geq N$ we will have the following for some arbitrary choice of positive ϵ

$$|s_n - s_m| < \epsilon.$$

Now let $s_m = s_{n-1}$ which from earlier gives

$$\implies |s_n - s_{n-1}| = |a_n| < \epsilon$$

$$\therefore |a_n - 0| < \epsilon.$$

Alternatively we directly take the limit of a_n

$$\lim a_n = \lim(s_n - s_{n-1})$$

and by the algebraic limit theorem for sequences it is understood that

$$\lim a_n = \lim s_n - \lim s_{n-1}.$$

Now s_n converges to some l by the summability of (a_n) and s_{n-1} must also converge to l given that it is a subsequence of s_n giving

$$\lim a_n = l - l = 0.$$

And so by both methods we have shown that

$$(a_n) \to 0.$$

$\square$

Remark 3.1.4. *It is good to note that this theorem is solely valid as a forwards implication and that as a backwards implication holds no water. Consider in fact Example 1 where the Harmonic Series was in question.*
It was shown at the start of Chapter 1 that the Harmonic sequence $\left(\frac{1}{n}\right)$ *is a null sequence and if the back implication were to hold then this would mean that the harmonic series is summable but as shown it in fact is not and gives an infinite sum !*

3.2 Series of Nonnegative Terms and Tests

Within this section, series consisting of terms which are non-negative shall be considered. This condition impacts the sequence of partial sums making it increasing. Within the two examples given at the start of chapter 3, we encountered such sequences and to show that they were convergent or divergent were able to consider whether they were bounded. This is a result of the monotone nature of the sequence of partial sums which thus allows for the use of the Monotone Convergence Theorem.

Boundedness Criterion
Theorem 3.2.1. *A series of nonnegative terms converges if and only if the sequence of partial sums is bounded above.*

Proof. For the forward implication we have that if a series is summable then its sequence of partial sums, (s_n), must converge and by **Theorem 2.3.1.** every convergent sequence is bounded and so ($\implies$) is valid.

For the backwards implication, the sequence of partial sums is monotone increasing since we're dealing with nonnegative series and by the premise of the implication this sequence is also bounded. By the Monotone Convergence Theorem for sequences (s_n) is thus convergent. By definition of summability of series, (a_n) must be summable by the convergence of (s_n). □

3.2.1 The Comparison Test

The comparison test allows for one to determine the convergence of some sequence knowing its relation to some series of known convergence in terms of order.
Theorem 3.2.2. *Assume (a_n) and (b_n) are sequences satisfying $0 \leq a_n \leq b_n \, \forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also.*

Proof. Let us first supply a proof by Cauchy's theorem.
Let the sequence of partial sums of (b_n) be labelled s_n :

$$s_n = b_1 + b_2 + \cdots + b_n.$$

Now since (b_n) is summable as $\sum_{n=1}^{\infty} b_n$ then (b_n) is convergent and that means that by the Cauchy Criterion there exists some $N \in \mathbb{N}$ such that for $n \geq m \geq N$ and an arbitrary choice of positive ϵ we have

$$|s_n - s_m| < \epsilon.$$

Consider that as in the technique applied in **Corollary 3.1.3.1.** for $m = n-1$ we have

$$|b_n| < \epsilon.$$

Now recall that by the premise $a_n \leq b_n$ which implies

$$|a_n| \leq |b_n| < \epsilon$$

$$\therefore \ |a_n| < \epsilon.$$

This means that (a_n) is convergent and by the definition of summability $\sum_{n=1}^{\infty} a_n$ is convergent.
It is good to note that this does not fall in to the category of the invalid back implication of **Corollary 3.1.3.1.** due to the use of the given order relation.

Now for the approach by Monotone Convergence.
Consider the sequences of partial sums

$$s_n = a_1 + \cdots + a_n \qquad\qquad t_n = b_1 + \cdots + b_n.$$

Now given that$\sum_{n=1}^{\infty} b_n$ is summable then we have that (t_n) is convergent and if it is convergent then it must be bounded by **Theorem 2.3.1.**.
Now also by premise

$$0 \leq s_n \leq t_n \,\forall n \in \mathbb{N}.$$

And since $\{t_n\}$ is bounded then $\exists\, M$ such that

$$\{t_n\} \leq M \,\forall n \in \mathbb{N}.$$

Using the inequality we have

$$s_n \leq t_n \leq M$$

which means s_n is bounded and monotone increasing given that term of the sequence are nonnoegative and by the monotone convergence theorem (s_m) is convergent and $\sum_{n=1}^{\infty} a_n$ is summable. □

Remark 3.2.3. *The contrapositive is an equivalent way to reformulate statements such that*

$$p \to q \equiv \neg q \to \neg p.$$

The contrapositive of the Comparison Test in the way it was stated allows for it to be reformulated in term of divergence such that;
Theorem 3.2.4. *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges also for* $0 \leq a_n \leq b_n \,\forall\, n \in \mathbb{N}$.

Example : - Comparison with Geometric Series Consider the rather bulky series

$$\sum_{n=1}^{\infty} \frac{2 + \sin^3(n+1)}{2^n + n^2}.$$

Determining whether this series is convergent may seem like a tall order but comparing it to a series of known convergence, typically the geometric series or the harmonic series, we are able to determine to the convergence of this series indirectly.
For this case in the denominator we have a self-multiplicative term which is reminiscent of a geometric series (to be introduced shortly), and so it readily evident that we can construct such a series out of the given series which would be of a larger size than it a convergent by nature of geometric series

$$0 \leq \frac{2 + \sin^3(n+1)}{2^n + n^2} \leq \frac{3}{2^n}.$$

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = 3 \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Examples implying the Limit Comparison Test Consider

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1 + \sin^2 n^3} \qquad \sum_{n=1}^{\infty} \frac{n+1}{n^2+1}.$$

Now both pf these sequences at for sufficiently large n look like sequences we are familiar with and whose convergence we understand

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \sum_{n=1}^{\infty} \frac{1}{n}.$$

meaning that the sequence on the left converges whilst that on the right diverges. This form of argument is formalised by the Limit Comparison test.

Limit Comparison Test

Theorem 3.2.5. *If $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c : c \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$.*

Proof. For ($\implies$);
By the premise it follows that by the definition of convergence $\exists\, N \in \mathbb{N} : \forall\, n \geq N$ we have that for some arbitrary choice of positive ϵ

$$\left|\frac{a_n}{b_n} - c\right| < \epsilon.$$

It is also given that the limit of this sequence is non-zero such that $c \neq 0$ and by **Theorem 2.2.1.** we know that the size of the ϵ-neighbourhood will be half the limit and so we may take $\epsilon = \frac{c}{2}$ giving

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}.$$

This may be restated be expanding the modulus to give

$$c - \frac{c}{2} < \frac{a_n}{b_n} < c + \frac{c}{2}$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$$

$$\therefore\ \frac{c}{2} < \frac{a_n}{b_n} \text{ and } \frac{a_n}{b_n} < \frac{3c}{2}.$$

Now we have that $\sum_{n=1}^{\infty} a_n$ converges by ($\implies$) so applying **Theorem 3.2.2 - The Comparison Test** given that $b_n < \frac{2a_n}{c}$ then $\sum_{n=1}^{\infty} b_n$ converges also.

For ($\impliedby$);
We instead consider $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{c}$ and proceed to follow a symmetric argument. $\square$

The Geometric Series A series is called geometric if it is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

Now we know that a series is summable if its sequence of partial sums is convergent. For $r = 1$ this can't be the case and we have divergence but consider $r \neq 1$ giving

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a(1 + r + r^2 + \cdots + r^{n-1}).$$

But the identity

$$(1-r)(1 + r + r^2 + \dots r^{n-1})$$

allows us to write

$$s_n = a(1 + r + r^2 + \cdots + r^{n-1}) = \frac{a(1-r^n)}{1-r}.$$

And so a geometric series is summable if

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

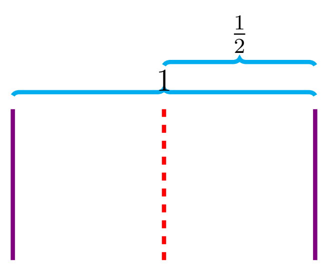
is convergent which by the algebraic limit theorem will give

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a\left(1 - \lim_{n\to\infty}(r^n)\right)}{1-r}.$$

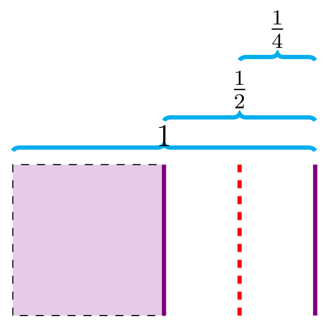
Now for $|r| < 1$, from **Example 2.5.3.** we saw that $\lim_{n\to\infty}(r^n) = 0$

$$\implies \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \text{ for } |r| < 1.$$

The Halving Sequence and answering the question At the start of the chapter a question was posed; if one were to split the distance between two objects in half and then repeat this process infinitely, would the objects meet ? Initially, one would be inclined to say no because there will always be some half of some half to add, but now that we have come to understand geometric sequences we can analyse to problem in this way.
So say we start with a unit distance 1 and split this;



Now we move to the new split distance and half that;



To investigate whether the objects will meet we must show that the distance covered by the object moving closer (the shaded purple area) on the left converges to the initial distance which separated them.
The object always covers half of the current separating distance giving a series which will look like

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

this series is nothing more than a geometric series with $r = \frac{1}{2}$ and $a = 1$! And given that $\left|\frac{1}{2}\right| < 1$ then as shown above, this sequence will converge such that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right) - 1 = \lim_{n\to\infty} s_n - 1 = \frac{1}{1-\frac{1}{2}} = 1.$$

And one is the initial separating distance ! So the objects will touch !

3.2.2 The Cauchy Condensation Test

The idea behind this test comes from the argument presented when we showed the divergence of the harmonic series. There we made the case that by considering a very particular subsequence of the sequence of partial sums, one could determine the nature of the sequence by just considering this subsequence. This subsequence is referred to as the condensed series
This test is typically used with series which have an n in the denominator.
Theorem 3.2.6. *For sequences, say (a_n) that are positive and decreasing and some definitions include null we have that*

$$\sum_{n=1}^{\infty} a_n \text{ is summable} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ is summable.}$$

So for a quick check let's consider the sequence that inspired this test, the harmonic series. So firstly we can use the test in this case since $\left(\frac{1}{n}\right)$ is both positive and decreasing now considering the sequence and its condensed form we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \qquad\qquad \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n}\right).$$

The condensed series reduces to the very divergent $\sum_{n=1}^{\infty} 1$ which so implies, correctly, the divergence of $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$. So on the face of it we have something that works, now how to prove it ?
Given that the decreasing nature is a prerequisite than using the assumed convergence of the condensed series we can show that both forms of the series are bounded and using the monotone convergence theorem will so show that the complete sequence is convergent.

Proof. For ($\Longleftarrow$).
Consider a positive, decreasing sequence (a_n) and let $\sum_{n=1}^{\infty} 2^n a_{2^n}$ be summable. By **Theorem 2.3.1.** $(2^n a_{2^n})$ is bounded and so some partial sum in this subsequence of partial sums, say t_k is bounded by some $M > 0$ for all $k \in \mathbb{N}$ such that we have

$$t_k = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k} \leq M.$$

Label the partial sums of (a_n) as s_m and for construction of the argument allow the choice of k to be sufficiently large enough to give

$$m \leq 2^{k+1} - 1.$$

$$\implies s_m \leq s_{2^{k+1}-1}.$$

With this inequality in hand let us look at $s_{2^{k+1}-1}$ more closely. Explicitly it may be written out as

$$s_{2^{k+1}-1} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \cdots + a_{2^k} + \ldots a_{2^{k+1}-1}.$$

So it appears that a subsequence from the partial sums of the condensed series is hidden within $s_{2^{k+1}-1}$. Now it is our job to bring a_{2^k} to the forefront so consider that since (a_n) is decreasing

$$s_{2^{k+1}-1} \leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + \cdots + a_{2^k} + \cdots + a_{2^k}.$$

But this nothing more than t_k !

$$t_k = a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + \cdots + a_{2^k} + \cdots + a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}.$$

And so we have $s_{2^{k+1}-1} \leq t_k$ which applying to the initial inequality

$$s_m \leq s_{2^{k+1}-1} \leq t_k.$$

But recall that t_k is bounded. Implying that $s_m \leq M \,\forall\, m \in \mathbb{N}$ also! Now s_m is constructed by terms of (a_m) and thus has decreasing terms which are now also bounded, thus by the monotone convergence theorem (s_m) is convergent which means that

$$\sum_{n=1}^{\infty} a_n \text{ is summable.}$$

For ($\implies$);
Assume that $\sum_{n=1}^{\infty} a_n$ is summable and so we have that the sequence of partial sums of this sequence is convergent. Now we allow the choice of k to be sufficiently small enough such that for a fixed choice of m we have

$$2^{k+1} - 1 \leq m.$$

$$\implies s_{2^{k+1}-1} \leq s_m.$$

Now exploring $s_{2^{k+1}-1}$ more closely we have

$$s_{2^{k+1}-1} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \cdots + a_{2^k} + \ldots a_{2^{k+1}-1}.$$

which is certainly greater than the following by the decreasing nature of the terms

$$s_{2^{k+1}-1} \geq \frac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1}a_{2^k}.$$

Which is nothing more than $\frac{1}{2}t_k$ meaning that within the inequality we now have

$$\frac{1}{2}t_k \leq s_{2^{k+1}-1} \leq s_m.$$

Now by the premise of ($\implies$) (s_m) is convergent by the summability of the series and so (s_m) is bounded by some $M \in \mathbb{N}$ giving that t_k is bounded as well for all $k \in \mathbb{N}$;

$$t_k \leq 2M.$$

Thus by the monotone convergence theorem (t_k) is convergent meaning that

$$\sum_{n=1}^{\infty} 2^n a_{2^n} \text{ is summable.}$$

$\square$

The Hyperharmonic Series Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^p} : p \neq 0.$$

We already know that for the case $p = 2$ this is convergent from the example on page 77 but let's generalise this result setting some bound for p that will serve as a condition for convergence.
Given that we have an n in the denominator let us employ the Cauchy Condensation test meaning that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is summable if and only if its corresponding condensed series is summable. This condensed series would be of form

$$\sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^{np}}\right) = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n}\right)^p = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)^{p-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n.$$

It is evident that this condensed series is geometric series with $a = 1$ and $r = \frac{1}{2^{p-1}}$. Now from pg.85 it is understood that for a geometric series to be convergent we need $|r| < 1$ thus consider that for the condensed series to be convergent we require

$$\left|\frac{1}{2^{p-1}}\right| < 1$$

and since $2^0 = 1$ we have

$$-(p-1) < 0$$
$$-p < -1$$
$$p > 1.$$

There by the Cauchy Condensation Test we have that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

3.2.3 Root Test

3.2.4 Ratio Test

3.2.5 Alternating Series Test

3.3 Types Of Convergence